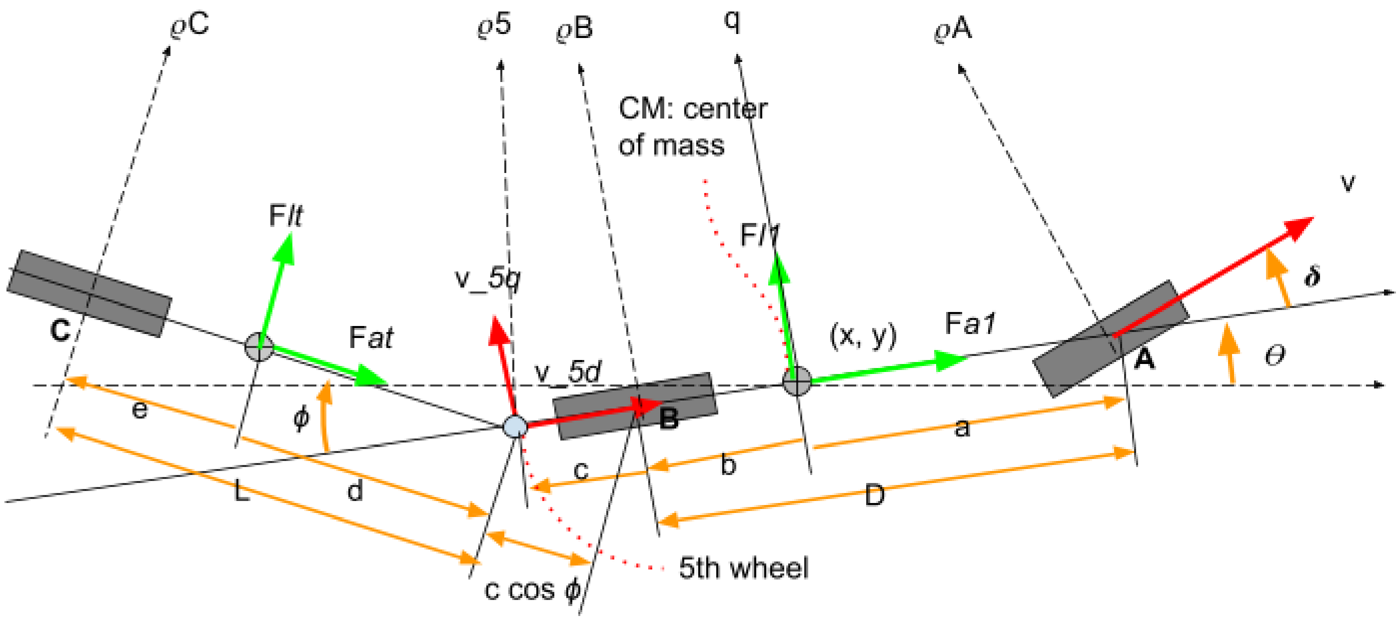
Bicycle model of an articulated tractor-trailer

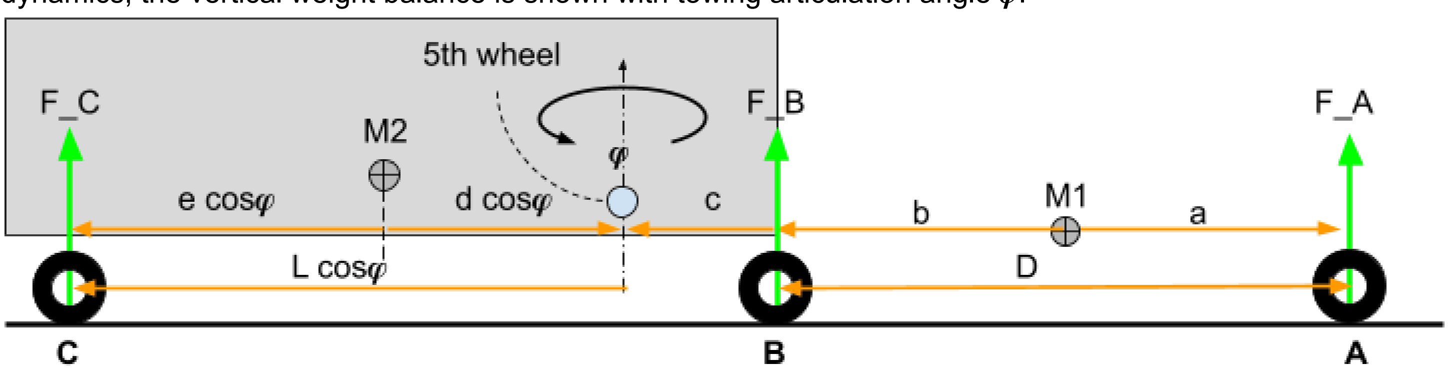
Here's a bicycle model of the typical tractor trailer coupled with a fifth wheel hitch, about which the trailer can only yaw with an angle ϕ in this planar model, so that the trailer's heading is $\theta - \phi$. In this diagram, the fifth wheel is behind the rear wheel of the tractor truck (c > 0), practical implementation keeps c < 0 (either right above or slightly in front of the truck's rear axle).



As in a simple 4-wheeled vehicle model, at low speed, a steering strategy keeps all wheels pointing at a common center point, with different radii: $\rho_A,\ \rho_B,\ \rho_C$ where $\rho_A \cos\delta = \rho_B$; $\rho_A \sin\delta = D = \rho_B \tan\delta$ But the relationship between $\rho_B,\ \rho_C$ takes a few more steps to derive. Firstly, draw the line ρ_5 from the 5th wheel to the same turning point. Its length is $\rho_5 = \sqrt{c^2 + \rho_B^2}$; $\rho_5 \cos\phi = \rho_C$; $L\tan\phi = \rho_C$, so given a desired ρ_C, we obtain $\phi = \tan^{-1}(\rho_C/L)$, $\rho_B = \sqrt{(\rho_C/\cos\phi)^2 - c^2}$, $\delta = \tan^{-1}(D/\rho_B)$

Static weight balance

As in the pickup truck case, the state variable (x, y) tracks the CM of the truck (M1). Neglecting the pitch dynamics, the vertical weight balance is shown with towing articulation angle φ.



The 5th wheel connects the trailer to the tractor truck, but allows roll, pitch, yaw rotation, so there is no net torque around the 5th wheel. The following force and moment balance results:

$$L\cos\phi\, F_C = m_2 g d\cos\phi \Rightarrow F_C = m_2 g d/L$$

$F_A + F_B + F_C = m_1 g + m_2 g \Rightarrow F_B = (m_1 g - F_A) + m_2 g e/L$

$cF_B + (c + D)F_A = m_1 g(b + c) \Rightarrow (c + D)F_A = m_1 g(b + c) - m_1 gc - m_2\ g\,ce/L + F_A c$

$\Rightarrow F_A = m_1 g\frac{b}{D} - m_2 g\ \frac{e}{L}\frac{c}{D};\ F_B\ = m_1 g\frac{a}{D} + m_2 g\frac{e}{L}\frac{c+D}{D};\ \ F_C = m_2 g\frac{d}{L}$

Note the elegant simplification of the forces when c = 0 (fifth wheel directly on top of the tractor's rear axle):

$F_A = m_1 g\frac{b}{D};\ F_B\ = m_1 g\frac{a}{D} + m_2 g\frac{e}{L};\ \ F_C = m_2 g\frac{d}{L}$

Note also the reduction in the normal force on the front (steering) axle when c > 0, increasing the possibility for losing steering traction in extreme case when $c >= \frac{m_1}{m_2}\ b\frac{L}{e}$

Dynamics

Rotating the $\hat{y}\hat{x}\hat{\theta}\hat{\phi}$ generalized axes by the heading angle Θ yields the body fixed (on the truck) coordinate frame $\hat{q}\hat{d}\hat{\theta}\hat{\phi}$, along which the forces and velocity can be calculated. For example, $\dot{x} = \cos(\theta)\ v_d - \sin(\theta)\ v_q$; $\dot{y} = \sin(\theta)\ v_d + \cos(\theta)\ v_q$. Forces act on the axles **{A, B, C}** and have the ***{l, a}*** or the ***{q, d}*** components. The ***q*** and the ***d*** subscripts mean the quadrature (lateral) and direct (axial) components, respectively. Sometimes I switch back and forth between the ***{l, a}*** and the ***{q, d}*** subscript (for the quadrature-direct). The axial force components are either driven, or approximated with a rolling resistance, while the lateral force is approximated with a tire model that considers the tire slip angle $\alpha_i = \tan^{-1}(v_{li}/v_{ai})$. The tractor's and the trailer's planar velocities are decomposed in the $\hat{q}\hat{d}$ frame: $\vec{v}_1 = v_d\ \hat{d} + v_q\ \hat{q}$,

$\vec{v}_2 = \{v_d\ + d\ \dot{\phi}\sin\phi\}\ \hat{d} + \{v_q\ + d\ \dot{\phi}\cos\phi\}\ \ \hat{q}$.

Using these speeds, the kinetic energy of the system consist of linear and rotational kinetic energies

$K(v_d, v_q, \dot{\theta}, \dot{\phi}) = \{m_1 \vec{v}_1^{\,T} \vec{v}_1\ +\ I_1 \dot{\theta}^2 + m_2 \vec{v}_2^{\,T} \vec{v}_2 +\ I_2(\dot{\theta} - \dot{\phi})^2\}/2$

$= \{m_1(v_d^{\,2} + v_q^{\,2})\ +\ I_1 \dot{\theta}^2\ +\ I_2(\dot{\theta}\ - \dot{\phi})^2 + m_2(v_d^{\,2} + v_q^{\,2} + d\dot{\phi}\ (d\dot{\phi}\ \ + 2\sin\phi\, v_d\ +\ 2\cos\phi\, v_q))\ \}/2$

If the truck is on a flat road, the potential energy P = 0. To get the Lagrangian of the K term, 2 derivatives are taken in each generalized coordinate q_j and then subtracted: $\frac{d}{dt}(\frac{\partial K}{\partial \dot{q}_j}) - \frac{\partial K}{\partial q_j}$.

$$\frac{\partial K}{\partial d} = \frac{\partial K}{\partial q} = \frac{\partial K}{\partial \theta} = 0; \frac{\partial K}{\partial \phi} = m_2\, d\dot{\phi}\ \ (\cos\phi\ v_d\ -\ \sin\phi\ v_q)$$

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{d}})\ = (m_1\ + m_2)\ \dot{v}_d + m_2\ d\sin\phi\ \ \ddot{\phi} + m_2\ d\cos\phi\dot{\phi}^2$$

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{q}})\ = (m_1\ + m_2)\ \dot{v}_q + m_2\ d\cos\phi\ \ \ddot{\phi} - m_2\ d\sin\phi\dot{\phi}^2$$

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{\theta}})\ = \frac{d}{dt}(I_1\dot{\theta} + I_2(\dot{\theta}\ \ - \dot{\phi}))\ = (I_1\ + I_2)\ \ddot{\theta} - I_2\ddot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}}\right) = m_2\, d(\sin\phi\, \dot{v}_d + \cos\phi\, \dot{v}_q) + (m_2 d^2 + I_2)\, \ddot{\phi} - I_2 \ddot{\theta} + m_2\, d\dot{\phi}\, (\cos\phi\, v_d - \sin\phi\, v_q)$$

Note that the last term is canceled out by $\frac{\partial K}{\partial \phi}$ so that

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}}\right) - \frac{\partial K}{\partial \phi} = m_2\, d\sin\phi\, \dot{v}_d + m_2\, d\cos\phi\, \dot{v}_q - I_2 \ddot{\theta} + m_2\, d(m_2 d^2 + I_2)\, \ddot{\phi}$$

Note that the terms separate into the inertial acceleration force (involving the time derivative of the speed) and centripetal force (involving the SQUARE of the angular rate) of the trailer yawing about the fifth wheel. That is:

$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = I\ddot{\vec{q}} + m_2 d\dot{\phi}^2 [\cos\phi,\ -\sin\phi, 0, 0]^T$ with the generalized inertia matrix I being

	$(m_1 + m_2)$	0	0	$m_2\, d\sin\phi$
	0	$(m_1 + m_2)$	0	$m_2\, d\cos\phi$
$I\ =$	0	0	$I_1 + I_2$	$-I_2$
	$m_2\, d\sin\phi$	$m_2\, d\cos\phi$	$-I_2$	$m_2 d^2 + I_2$

This positive definite symmetric matrix is readily invertible using the denominator $den = m_1 m_2 d^2 (I_1 + I_2)$, the shorthand $m_t = m_1 + m_2$, $I_t = I_1 + I_2$, the inverse (which is also symmetric) is

	$I_t m_2 d^2 (1 - m_2/m_t \cos^2\phi) + I_1 I_2$	$m_2^{\,2} d^2 \cos\phi\ \sin\phi\ I_t/m_t$	$-I_2 m_2\, d\sin\phi$	$-I_t m_2\, d\sin\phi$
$(m_t I_1 I_2 + m_1 m_2\, d^2 I_t)$	symmetric	$I_t m_2 d^2 (1 - m_2/m_t \sin^2\phi) + I_1 I_2$	$-I_2 m_2\, d\cos\phi$	$-I_t m_2\, d\cos\phi$
$\bullet\, I^{-1} =$	symmetric	symmetric	$m_t I_2 + m_1 m_2\ d^2$	$m_t I_2$
	symmetric	symmetric	symmetric	$m_t I_t$

Where the symmetric terms are omitted for clarity. Note that the ratio $m_2/m_t < 1$

The velocity of each axle is derived by superposition of the gradient of the axle position (the Jacobian J), times the coordinate change rate q_j, i.e. $\vec{v}_i = [v_d\, v_q\, \dot{\theta}\ \dot{\phi}]^T = \sum_{j \in \{d, q, \theta, \phi\}} \frac{\partial \vec{p}_i}{\partial q_j} \dot{q}_j = J\dot{\vec{q}}$. We consider the positions $\vec{p}_{\{A, B, C\}}$ where external forces act, and their partial derivatives relative to the generalized coordinates $q\ \in \{d, q, \theta, \phi\}$.

$$\frac{\partial \vec{p}_A}{\partial q} = [\hat{d},\ \hat{q},\ a\hat{q},\ 0]^T,$$

$$\frac{\partial \vec{p}_B}{\partial q} = [\hat{d},\ \hat{q},\ -b\hat{q},\ 0]^T,$$

$$\frac{\partial \vec{p}_C}{\partial q} = [\hat{d},\ \hat{q},\ -L\ \sin\phi\, \hat{d} - \{L\ \cos\phi + b + c\}\, \hat{q},\ L\ \sin\phi\, \hat{d} + L\ \cos\phi\, \hat{q}]^T$$

Note that unlike d, which varies with the loading condition, $d + e = L$ is a design time constant.

Substituting the terms into the summation above, the velocities are determined to:

$\vec{v}_A = v_d \hat{d} + (v_q + a\dot{\theta})\hat{q}$

$\vec{v}_B = v_d \hat{d} + (v_q - b\dot{\theta})\hat{q}$ $\vec{v}_C = \{v_d - L\sin\phi\,\dot{\theta} + L\sin\phi\,\dot{\phi}\}\hat{d} + \{v_q - (b + c + L\cos\phi)\,\dot{\theta} + L\cos\phi\,\dot{\phi}\}\hat{q}$

The generalized for is force on each axle multiplied by the Jacobian above: $\vec{Q}_{i\in\{d,q,\theta,\phi\}} = \sum_{j\in\{A,B,C\}} \frac{\partial \vec{p}_i}{\partial q_j} F_j = J\vec{F}$

$\vec{F}_A = \{(F_{dA} - F_{bA})\cos\delta - F_{qA}\sin\delta\}\hat{d} + \{(F_{dA} - F_{bA})\sin\delta + F_{qA}\cos\delta\}\hat{q}$

$\vec{F}_B = (F_{dB} - F_{bB})\hat{d} + F_{qB}\hat{q}$

$\vec{F}_C = \{(F_{dC} - F_{bC})\cos\phi + F_{qC}\sin\phi\}\hat{d} + \{(F_{bC} - F_{dC})\sin\phi + F_{qC}\cos\phi\}\hat{q}$

Multiplication by the Jacobian yields

$Q_d = (F_{dA} - F_{bA})\cos\delta - F_{qA}\sin\delta + (F_{dB} - F_{bB}) + (F_{dC} - F_{bC})\cos\phi + F_{qC}\sin\phi$

$Q_q = (F_{dA} - F_{bA})\sin\delta + F_{qA}\cos\delta + F_{qB} + (F_{dC} - F_{bC})\sin\phi + F_{qC}\cos\phi$

$Q_\theta = a\{(F_{dA} - F_{bA})\cos\delta + F_{qA}\sin\delta\} - b(F_{dB} - F_{bB}) + (b + c)\sin\phi\,(F_{dC} - F_{bC})\ \cos\phi - \{L + (b + c)\cos\phi\}F_{qC}$

$Q_\phi = LF_{qC}$

Where $\{F_{bA}, F_{bB}, F_{bC}\}$ are the resistive forces at the wheels $\{A, B, C\}$.

Balancing the generalized inertial force to the generalized external force, a state equation is obtained:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = I\ddot{\vec{q}} + m_2 d\dot{\phi}^2[\cos\phi, \sin\phi, 0,\ 0]^T = \vec{Q}_i = \sum_{j\in\{A,B,C\}} \frac{\partial \vec{p}_i}{\partial q_j} F_j = J\vec{F}$$

$\dot{\vec{V}} = [\dot{d}\ \dot{q}\ \dot{\theta}\ \dot{\phi}]^T = I^{-1}\{\vec{Q} - m_2 d\dot{\phi}^2[\cos\phi\ \sin\phi\ 0\ 0]^T\}$

The forces on the axle (wheel, really) are: lateral force determined by the tire model, and the rolling friction.

Kinematics

Control algorithm to track reference path

When driving forward at moderate speed, the same control algorithm described previously will work fine. But when parking in a tight spot, controlling both the trailer and the truck pose is important. If same approach is used, then the articulation angle φ should first be controlled, and the steering angle δ can be controlled in a inner loop. MPC is probably more appropriate.

Rear wheel drive, viewing the rear axle position and velocity

Non-holonomic constraint at the front axle A ⇒ $D\dot{\theta} = v\ \tan\delta$, so $\dot{\theta} = v\ \tan\delta/D$ and

$\ddot{\theta} = (\dot{v}\tan\delta + v\dot{\delta}/\cos^2\delta)/D$

The 5th wheel has a velocity in d-q frame: $v_{5d}\,\hat{d} + v_{5q}\,\hat{q} = v\,\hat{d} - (c/D)\,v\tan\delta\,\hat{q}$. Note the negative sign on the $\hat{q}$ component is due to the 5th wheel being behind the rear axle B. Rotate this vector by the articulation angle ϕ to resolve in the trailer's frame's axial and lateral directions:

$$v_{5d}\cos\phi - v_{5q}\sin\phi = v\cos\phi + (c/D)\,v\tan\delta\sin\phi$$
$$v_{5d}\sin\phi + v_{5q}\cos\phi = v\sin\phi - (c/D)\,v\tan\delta\cos\phi$$

The non-holonomic constraint at C means that this the axial direction is unconstrained, but the lateral component is only due to the rotation about C ⇒ $L(\dot{\theta} - \dot{\phi}) = L\dot{\theta}_C = v\{\sin\phi - (c/D)\ \tan\delta\cos\phi\}$

Note that the trailer's yaw rate $\dot{\theta}_C = \dot{\theta} - \dot{\phi}$ and articulation angle is largely controlled by the current articulation angle φ: $\dot{\theta}_C = v\{\sin\phi - (c/D)\ \tan\delta\cos\phi\}/L$

$$\begin{aligned}\Rightarrow \dot{\phi} &= \dot{\theta} - v\{\sin\phi - (c/D)\ \tan\delta\cos\phi\}/L \\ &= v\,(\tan\delta/D - \{\sin\phi - (c/D)\ \tan\delta\cos\phi\}/L) \\ &= v\,(\{1 + (c/L)\cos\phi\}\tan\delta/D - \sin\phi/L) \\ &= (-v)\ (\sin\phi/L - \{1 + (c/L)\cos\phi\}\tan\delta/D)\end{aligned}$$

The 2nd expression is helpful when backing up, i.e. v < 0. It shows that backing up a trailer is like trying to balance an inverted pendulum; beyond a certain ϕ, it may be impossible to recover without first reversing the gear to the forward.

For the articulation angle acceleration, take the time derivative of $\dot{\phi}$:

$$\begin{aligned}\ddot{\phi} &= \dot{v}\,(\{1 + (c/L)\cos\phi\}\tan\delta/D - \sin\phi/L) \\ &\quad + v\{(c/D)\,\dot{\delta}\cos\phi/\cos^2\delta - (c/D)\,\dot{\phi}\sin\phi\tan\delta - \dot{\phi}\cos\phi\}/L \\ &= \dot{v}\,(\{1 + (c/L)\cos\phi\}\tan\delta/D - \sin\phi/L) \\ &\quad + \frac{1}{D\cos^2\delta}v\dot{\delta} + \frac{c\cos\phi}{LD\cos^2\delta}v\dot{\delta} - \frac{c\sin\phi\tan\delta}{LD}v\dot{\phi} - \frac{\sin\phi}{L}v\,\dot{\phi} \\ &= \dot{v}\frac{(L + c\cos\phi)\tan\delta - L\sin\phi}{LD} + \frac{L + c\cos\phi}{LD\cos^2\delta}v\dot{\delta} - \frac{(D + c\tan\delta)\sin\phi}{LD}v\dot{\phi}\end{aligned}$$

The truck's rear wheel (on axle B) spins with tangential speed *v*, but neither the axles B and C are slipping in lateral direction (due to the non-holonomic constraint), so the their velocities can be written:

$$\vec{v}_A = v\,\hat{d} + v\ \tan\delta\,\hat{q}$$

$$\vec{v} = v\,\hat{d} + (b/D)v\ \tan\delta\,\hat{q}$$

$$\vec{v}_B = v\,\hat{d}$$

$$\vec{v}_C = \vec{v}_5 - L(\dot{\theta} - \dot{\phi})\,(\sin\ \phi\,\hat{d} + \cos\phi)\,\hat{q} = \{v - L(\dot{\theta} - \dot{\phi})\ \sin\ \phi\}\,\hat{d} + \{(b/D)v\ \tan\delta - L(\dot{\theta} - \dot{\phi})\ \sin\ \phi\}\,\hat{q}$$

The state vector now omits the lateral speed at the axle B, which should be 0 (non-holonomic constraint assumption) $\vec{X} = [x_B\, y_B\, \theta\, \phi\, v_{Bd}\, \dot{\theta}\, \dot{\phi}]^T$, and its derivative

$$\dot{\vec{X}}_B = [v\cos\theta,\ v\sin\theta,\ v\frac{\tan\delta}{D},\ v\frac{\tan\delta}{D} - v\,(\frac{c\cos\phi}{L}\frac{\tan\delta}{D} - \frac{\sin\phi}{L}),\ \ldots$$

$$\dot{v},\ \frac{\tan\delta}{D}\dot{v} + \frac{1}{D\cos^2\delta}v\dot{\delta}, \frac{(L+c\cos\phi)\tan\delta - L\sin\phi}{LD}\dot{v} + \frac{L+c\cos\phi}{LD\cos^2\delta}v\dot{\delta} - \frac{(D+c\tan\delta)\sin\phi}{LD}v\dot{\phi}]^T$$

The position and velocities of the axles A, C, and the CM can be worked out from $\vec{X}_B$.

$$\vec{x}_A = (x_B + D\cos\theta)\ \hat{x} + (y_B + D\sin\theta)\ \hat{y}$$

$$\vec{x} = (x_B + b\cos\theta)\ \hat{x} + (y_B + b\sin\theta)\ \hat{y}$$

$$\vec{x}_C = \{x_B - c\cos\theta - L\cos(\theta - \phi)\}\ \hat{x} + \{y_B - c\sin\theta - L\sin(\theta - \phi)\}\ \hat{y}$$

Sanity check: tractor trailer

As a sanity check, consider when the 5th wheel is almost directly on top of the rear axle (as in a commercial tractor trailer), i.e. c = 0.

$$L(\dot{\theta} - \dot{\phi}) = L\dot{\theta}_C = v\{\sin\phi\}$$

$$\dot{\phi} = v(\tan\delta/D - \sin\phi/L) = (-v)\ (\sin\phi/L - \tan\delta/D)$$

$$\dot{\vec{X}}_B = [v\cos\theta,\ v\sin\theta,\ v\frac{\tan\delta}{D},\ v(\frac{\tan\delta}{D} + \frac{\sin\phi}{L}),\ \ldots$$

$$\dot{v},\ \frac{\tan\delta}{D}\dot{v} + \frac{v}{D\cos^2\delta}\dot{\delta}, \frac{\tan\delta - \sin\phi}{D}\dot{v} + \frac{v}{D\cos^2\delta}\dot{\delta} - \frac{\sin\phi\, v}{L}\dot{\phi}]^T$$

$$\vec{x}_C = \{x_B - L\cos(\theta - \phi)\}\ \hat{x} + \{y_B - L\sin(\theta - \phi)\}\ \hat{y}$$

Front wheel drive kinematics

The truck's front wheel (on axle A) spins with tangential speed *v* and steering angle δ from the $\hat{d}$ axis. The axial speed is $v_{Bd} = v\cos\delta\ \hat{d}$

Non-holonomic constraint at the front axle A ⇒ $D\dot{\theta} = v\ \sin\delta$, so $\ddot{\theta} = (\dot{v}\sin\delta + v\dot{\delta}\cos\delta)/D$

Non-holonomic constraint at the trailer axle C means that the trailer is rotating about C due to the lateral speed resolved at the 5th wheel, which is a vector $v_{5d}\hat{d} + v_{5q}\hat{q} = v\cos\delta\,\hat{d} - (c/D)\,v\sin\delta\,\hat{q}$

Rotate this vector by the articulation angle φ to resolve in the trailer's frame's axial and lateral direction:

$$v_{5d}\cos\phi - v_{5q}\sin\phi = v\cos\delta\ \cos\phi + (c/D)\,v\sin\delta\sin\phi$$

$$v_{5d}\sin\phi + v_{5q}\cos\phi = v\cos\delta\ \sin\phi - (c/D)\,v\sin\delta\cos\phi$$

The non-holonomic constraint at C means that this lateral velocity component is only due to the rotation about C ⇒ $L(\dot{\theta} - \dot{\phi}) = L\dot{\theta}_C = v\{\cos\delta\ \sin\phi - (c/D)\ \sin\delta\cos\phi\}$

Note that the trailer's heading $\theta_C = \theta - \phi$ and articulation angle is largely controlled by the current articulation angle φ: $\dot{\theta}_C = v\{\cos\delta\ \sin\phi - (c/D)\ \sin\delta\cos\phi\}\,/\,L$

$$\Rightarrow \dot{\phi} = \dot{\theta} - v\{\cos\delta\ \sin\phi - (c/D)\sin\delta\cos\phi\}\,/\,L$$
$$= v\{\sin\delta/D - \cos\delta\ \sin\phi/L + c\sin\delta\cos\phi/DL\}$$

$$\dot{\phi} = v\{(1 + c\cos\phi/L)\sin\delta/D - \cos\delta\ \sin\phi/L\} = (-v)\{\cos\delta\ \sin\phi/L - (1 + c\cos\phi/L)\sin\delta/D\}$$

$$= v\{\frac{L+c\cos\phi}{LD}\sin\delta - \frac{\sin\phi}{L}\cos\delta\} = (-v)\{\frac{\sin\phi}{L}\cos\delta - \frac{L+c\cos\phi}{LD}\sin\delta\}$$

The last expression is helpful when backing up, i.e. $v < 0$. It shows that backing up a trailer is like trying to balance an inverted pendulum.

Taking the time derivative of $\dot{\phi}$:

$$\ddot{\phi} = \dot{v}\,\{(1 + c\cos\phi/L)\sin\delta/D - \cos\delta\,\sin\phi/L\}$$
$$+\; v\,\{\dot{\delta}\,(1 + c\cos\phi/L)\cos\delta/D - \dot{\phi}\,c\sin\phi\sin\delta/LD + \dot{\delta}\,\sin\delta\,\sin\phi/L - \dot{\phi}\cos\delta\,\cos\phi/L\}$$
$$= (\dot{v}/LD)\,\{(LD + cD\cos\phi)\sin\delta - D\cos\delta\,\sin\phi\}$$
$$+\; (v/LD)\,\{\dot{\delta}\,(LD + c\cos\phi\cos\delta + D\sin\phi\sin\delta) - \dot{\phi}\,(c\sin\phi\sin\delta + D\cos\phi\cos\delta)\}$$

The state vector now omits the lateral speed at the axle B, which should be 0 (non-holonomic constraint assumption) $\vec{X}_B = [x_B\, y_B\, \theta\, \phi\, v_{Bd}\, \dot{\theta}\, \dot{\phi}]^T$, and its derivative

$$\dot{\vec{X}}_B = [v\ \cos\delta\cos\theta,\ v\ \cos\delta\sin\theta, \frac{v\sin\delta}{D},\ v\,\{\frac{L+c\cos\phi}{LD}\sin\delta - \frac{\sin\phi}{L}\cos\delta\},\ \ldots$$
$$\dot{v}\cos\delta - \dot{\delta}v\sin\delta,\ (\dot{v}\sin\delta + v\,\dot{\delta}\cos\delta)/D,\ \ldots$$
$$(\dot{v}/LD)\,\{(LD + cD\cos\phi)\sin\delta - D\cos\delta\,\sin\phi\}\ \ldots$$
$$+\; (v/LD)\,\{\dot{\delta}\,(LD + c\cos\phi\cos\delta + D\sin\phi\sin\delta) - \dot{\phi}\,(c\sin\phi\sin\delta + D\cos\phi\cos\delta)\}]^T$$

The position and velocities of the axles A, C, and the CM can be worked out from $\vec{X}_B$.

$$\vec{x}_A = (x_B + D\cos\theta)\ \hat{x} + (y_B + D\sin\theta)\ \hat{y}$$

$$\vec{x} = (x_B + b\cos\theta)\ \hat{x} + (y_B + b\sin\theta)\ \hat{y}$$

$$\vec{x}_C = \{x_B - c\cos\theta - L\cos(\theta - \phi)\}\ \hat{x} + \{y_B - c\sin\theta - L\sin(\theta - \phi)\}\ \hat{y}$$

$$\vec{v}_A = v\cos\delta\ \hat{d} + \{v\sin\delta\ + D\dot{\theta}\}\ \hat{q},$$

$$\vec{v}_B = v\cos\delta\ \hat{d},$$

$$\vec{v} = v\ \cos\delta\,\hat{d} + (b/D)v\ \sin\delta\,\hat{q},$$

$$\vec{v}_C = \vec{v}_5 - L(\dot{\theta} - \dot{\phi})\sin\ \phi\ \hat{d}\ - L(\dot{\theta} - \dot{\phi})\cos\phi\ \hat{q}$$

Sanity check: tractor trailer

As a sanity check, consider when the 5th wheel is almost directly on top of the rear axle (as in a commercial tractor trailer), i.e. c = 0.

$$L(\dot{\theta} - \dot{\phi}) = L\dot{\theta}_C = v\cos\delta\ \sin\phi$$

$$\dot{\phi} = v\,\{\sin\delta/D - \sin\phi\cos\delta/L\} = (-\,v)\,\{\sin\phi\cos\delta/L - \sin\delta/D\}$$

$$\dot{\vec{X}}_B = [v\ \cos\delta\cos\theta,\ v\ \cos\delta\sin\theta, \frac{v\sin\delta}{D},\ v\,\{\frac{\sin\delta}{D} - \frac{\sin\phi\cos\delta}{L}\},\ \ldots$$
$$\dot{v}\cos\delta - \dot{\delta}v\sin\delta,\ (\dot{v}\sin\delta + v\,\dot{\delta}\cos\delta)/D,\ \ldots$$
$$(\dot{v}/L)\,\{L\sin\delta - D\cos\delta\ \sin\phi\} + (v/L)\,\{\dot{\delta}\,(L + \sin\phi\sin\delta) - \dot{\phi}\ \cos\phi\cos\delta\}\,]^T$$

$$\vec{x}_C = \{x_B - L\cos(\theta - \phi)\}\ \hat{x} + \{y_B - L\sin(\theta - \phi)\}\ \hat{y}$$

$$\vec{v}_C = v\,\{\cos^2\phi\}\ \hat{d}\ + v\,\{b\tan\delta/D - \sin\phi\cos\phi\}\ \hat{q}$$