Road Vehicle Dynamics

Problems and Solutions

Rao V. Dukkipati
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Contents

Preface vii

Chapter 1 Introduction.. 1

Chapter 2 Analysis of Dynamic Systems 43

Chapter 3 Tire Dynamics.. 89

Chapter 4 Ride Dynamics 117

Chapter 5 Vehicle Rollover Analysis 131

Chapter 6 Handling Dynamics 143

Chapter 7 Braking 159

Chapter 8 Acceleration 189

Chapter 9 Total Vehicle Dynamics 219

Chapter 10 Accident Reconstruction............................... 243

Appendix A Vector Algebra 313

Appendix B Matrix Analysis....................................... 321

Appendix C Laplace Transforms.................................... 367

Appendix D Glossary of Terms 419

Appendix E Direct Numerical Integration Methods.................. 427

Appendix F Units and Conversions 493

Appendix G Accident Reconstruction Formulae...................... 495

Preface

Objective of This Book

Road Vehicle Dynamics: Problems and Solutions is a comprehensive supplement to our book, *Road Vehicle Dynamics* (R-366), which was published by SAE International in 2008. More than 300 example problems and solutions are presented in an organized and systematic manner to offer detailed explanations of complex aspects of this exciting, challenging, and multidisciplinary field.

Road Vehicle Dynamics: Problems and Solutions presents vehicle dynamics methods based on mathematical fundamentals and stresses physical system modeling. The book is organized around the concepts of road vehicle dynamics as they have been developed in the frequency and time domain for an introductory undergraduate or graduate course for engineering students of all disciplines.

Generally speaking, road vehicle dynamics can seem to be a difficult subject to understand and learn. Despite the availability of a few textbooks in this field, students continue to remain perplexed because of the outcomes of the numerous conditions that often must be kept in mind and correlated when solving a given problem. In addition, different possible interpretations of the terms used in road vehicle dynamics can contribute to the difficulties experienced by students.

The main objective of this book is to provide readers with the opportunity to improve their problem-solving skills by using systematic rules of analysis that may be followed in a step-by-step manner. In schools, students often experience difficulty in understanding and learning road vehicle dynamics for the following reasons:

- No systematic rules of analysis that may be followed in a step-by-step manner have been formulated or developed to find solutions to the problems at hand.
- Most available textbooks on the subject usually explain a given principle in an abstract manner, leaving students confused about the application of that principle.
- Too few examples are provided in textbooks, and they often do not provide sufficient bases for students to solve the problems in their homework assignments or examinations.
- The examples presented in textbooks often are difficult to understand.
- Examples often do not include diagrams/graphs, where appropriate.
- Students often spend too many hours solving a single problem, sometimes by simple guesswork or by trial and error.

We believe that road vehicle dynamics is a subject that is best learned by allowing individuals to review on their own the methods of analysis and solution techniques. This method of learning is similar to that practiced in various scientific laboratories and in medical fields.

The objective of this book is to introduce individuals from a variety of disciplines and backgrounds to the vast array of problems that are amenable to numerical solution in road vehicle dynamics. Emphasis is placed on application rather than on pure theory, which, although kept to a minimum, is presented in mostly a heuristic and intuitive manner. This is deemed sufficient for individuals to fully understand the workings, efficiencies, and shortcomings or failings of each technique. Because we intended this book as a first course on road vehicle dynamics, the concepts have been applied in simple terms, and the solution procedures have been explained in detail.

Audience

This book is intended for vehicle designers, developers, and evaluators, as well as senior undergraduate students and/or graduate students. No previous knowledge of road vehicle dynamics is assumed. This book is appropriate for several groups of audiences, including the following.

- Senior undergraduate and graduate students in mathematics, science, and engineering who are taking an introductory course on road vehicle dynamics will find the book helpful in understanding the subject.
- The book can be adapted for a short professional course on road vehicle dynamics.
- Design and research engineers will be able to draw upon the book in selecting and developing road vehicle dynamics for analytical and design purposes.
- Practicing engineers and managers will be able to learn more about the basic principles and concepts involved in road vehicle dynamics and how these can be applied to address their own workplace concerns.

Content

This book consists of ten chapters and an extended list of appendices. The chapters include an introduction, analysis of dynamic systems, vehicle forces and tire mechanics, ride dynamics, roll dynamics, handling and steering, braking, accelerating, total vehicle dynamics, and accident reconstruction. The appendices span vector and matrix algebra, Fourier series, Laplace transformation, vehicle dynamics terminology, direct numerical integration methods, conversion of units, and accident reconstruction formulae. In this book, we pay particular attention to the issue of safety. In fact, safety considerations are included in most chapters of this book. In addition, two complete chapters are devoted to roll dynamics and accident reconstruction to address the legal issues that may result from an automotive accident.

Numerous worked examples of problems and solutions offer detailed explanations and guide readers through each set of problems to enable them to save a great deal of time and effort in arriving at an understanding of problems in road vehicle dynamics.

This book is intended to help students of road vehicle dynamics find their way through complex material involving a diverse variety of concepts. It offers detailed illustrations of solution methods that may not be clearly apparent; representative and typical problems given in homework, class work, and examinations; step-by-step explanations; and the opportunity to save time and effort in arriving at an understanding of problems in road vehicle dynamics. These worked-out example problems and solutions should be of interest to a wide audience, including students, vehicle designers, developers, and evaluators.

Chapter 1

Introduction

Problem 1.1

For the vibrating system shown in Figure 1.1:

a. Determine the number of degrees of freedom needed to specify the motion of the system.

b. Identify a set of generalized coordinates.

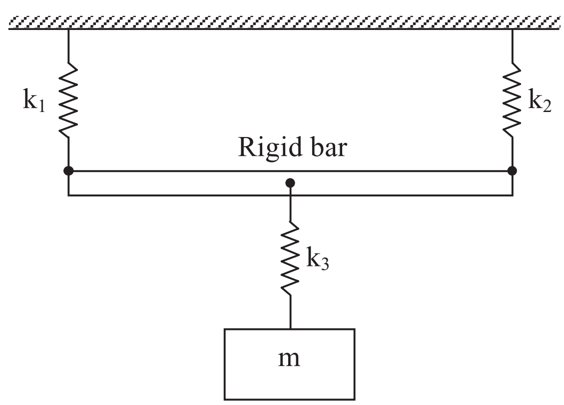

Figure 1.1 Vibrating system.

Solution to Problem 1.1

a. Here, the rotational and translational motions of the rigid bar are independent of each other. Also the motion of the hanging mass m is an independent motion. Hence, the system has three degrees of freedom.

b. The choice of generalized coordinates is not unique, and several sets of possible choices exist, as shown in Figures 1.1(a) through (c).

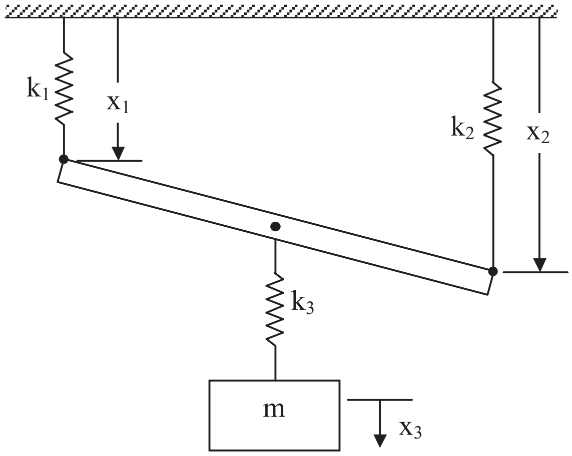

Figure 1.1 (a) One possible free body diagram.

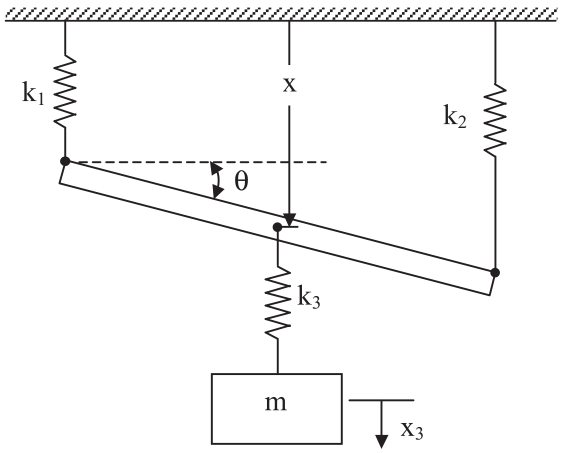

Figure 1.1 (b) Another free body diagram.

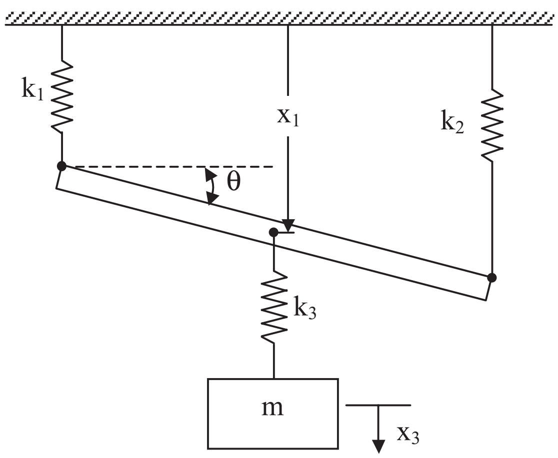

Figure 1.1 (c) A third free body diagram.

Problem 1.2

For the vibrating system shown in Figure 1.2:

a. Determine the number of degrees of freedom needed to specify the motion of the system.

b. Identify a set of generalized coordinates.

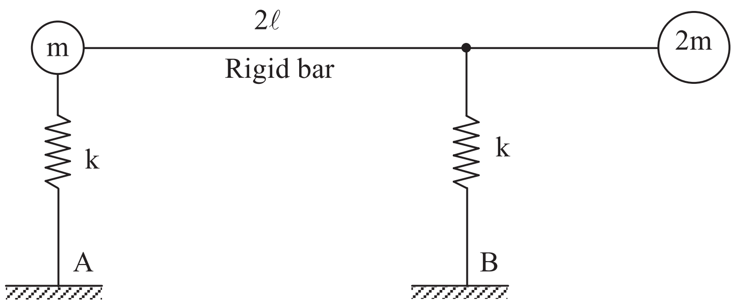

Figure 1.2 Vibrating system.

Solution to Problem 1.2

a. The system shown in Figure 1.2 has two degrees of freedom.

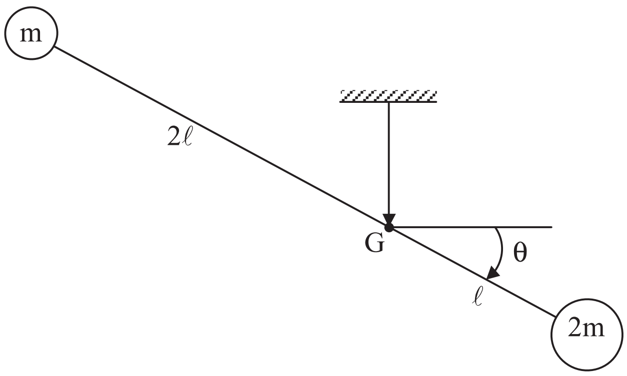

Figure 1.2 (a) Free body diagram.

b. One choice of a set of generalized coordinates is θ, which is the clockwise angular displacement of the rigid bar from the equilibrium position of the system, and x, which is the downward displacement of point G from the equilibrium position of the system, as shown in Figure 1.2.

Problem 1.3

Develop a mathematical model of a washing machine as a lumped-parameter or discrete parameter system. Consider a washing machine standing on elastomeric (rubber) mounts, with the drum rotating in the vertical plane with constant angular acceleration. Also assume that all components or elements of the washing machine undergo no elastic deformations.

Solution to Problem 1.3

Figure 1.3(a) shows a washing machine that is mounted on rubber pads. The drum is assumed to be rotating in a vertical plane relative to the body of the machine with a constant angular velocity. The following assumptions can be made:

a. The body of the machine and the drum do not undergo elastic deformations.

b. The clothes are distributed uniformly around the drum.

c. Inertia properties remain unchanged with time.

Figure 1.3(b) shows the mathematical model of the washing machine, where the mass m represents the combined mass of the body of the machine, the drum, and the clothes. Also, x(t) represents the vertical displacement of mass m. The rubber pads can act in parallel as springs and dashpots. Assuming symmetry of the machine, the spring constants and the viscous damping coefficients on the left and right supports are represented by $k/2$ and $c/2$, respectively.

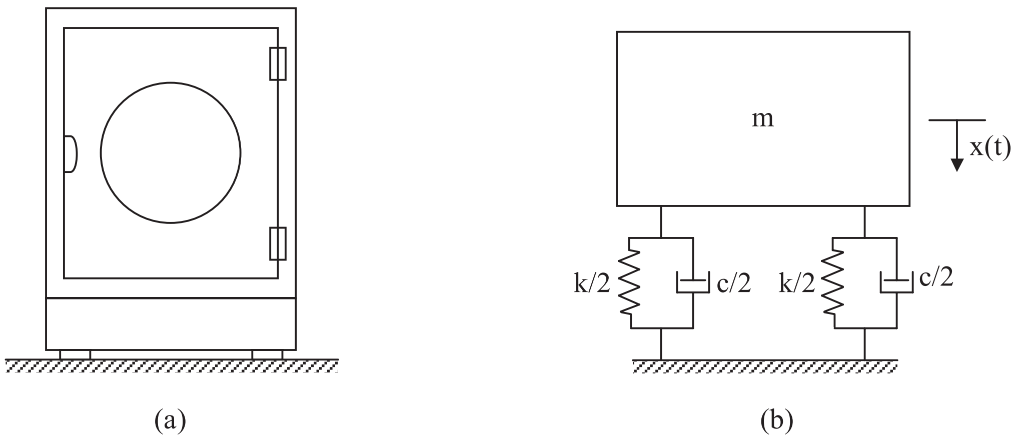

Figure 1.3 (a) A washing machine, and (b) a mathematical model of the washing machine.

Problem 1.4

Develop a mathematical model of a motorcycle with a rider for use in investigating vibration in the vertical direction. The elasticity of the tires, elasticity and viscous damping of the struts, wheel masses, mass, elasticity, and damping of the rider are to be considered in the model.

Solution to Problem 1.4

Let

m_{eq} = equivalent mass of the system (wheels, motorcycle, and rider)

k_{eq} = equivalent stiffness of the system (tires, struts, and rider)

c_{eq} = equivalent damping of the system (struts and rider)

k_s = stiffness of the strut

c_s = damping of the strut

m_c = mass of the motorcycle

m_R = mass of the rider

k_t = stiffness of the tirew

m_w = mass of the wheel

If we consider the equivalent values for mass, stiffness, and damping, then we see that a single degree of freedom model can be modeled as shown in Figure1.4(b). The masses of the wheels, elasticity of the tires, and elasticity and damping of the struts separately result in a model that can be represented as shown in Figure 1.4(c). Similarly, when the elasticity and the damping of the rider are considered, the resulting model is as shown in Figure 1.4(d). If one combines the spring constants of the front and rear tires and the masses of the front and rear wheels, and if the spring and damping constants of both struts are single quantities, then the model shown in Figure 1.4(e) is obtained.

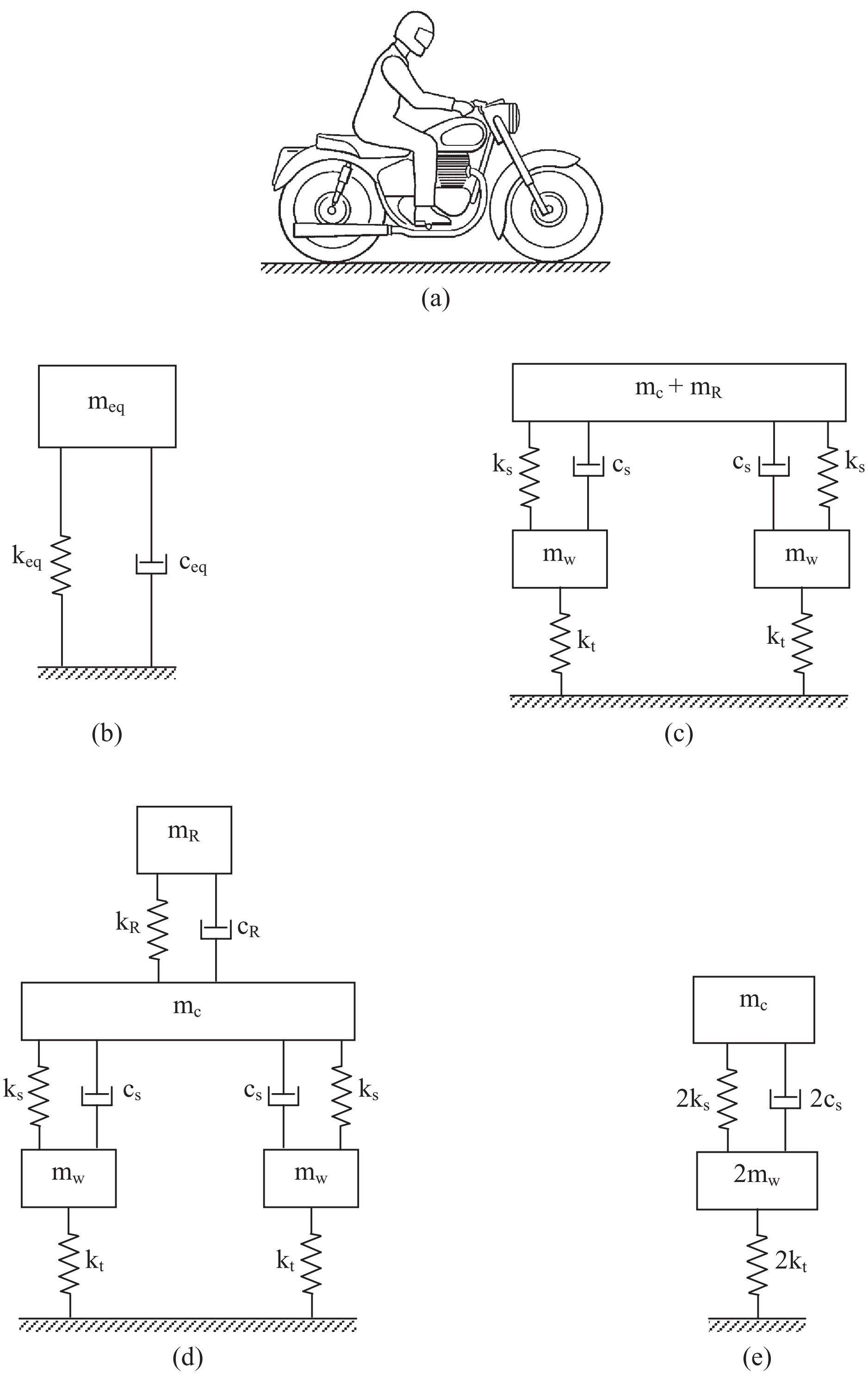

Figure 1.4 Motorcycle with a rider, and the equivalent mathematical models: (a) motorcycle with a rider; (b) single-degree-of-freedom model; (c) two-degrees-of-freedom model; (d) three-degrees-of-freedom model; and (e) another two-degrees-of-freedom model.

Problem 1.5

Develop a mathematical model of a multistory building, as shown in Figure 1.5.

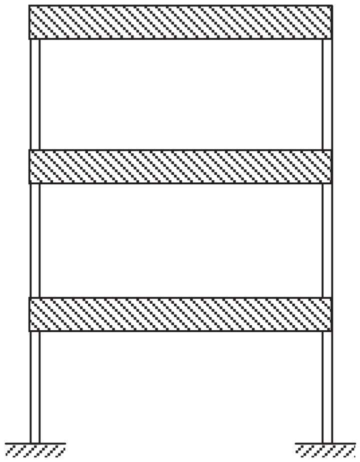

Figure 1.5 Multistory building.

Solution to Problem 1.5

If it is assumed that the mass of the frame of the building is small compared with the mass of the floor, then the system shown in Figure 1.5 can be represented by the multiple-degrees-of-freedom vibrating system in the vertical plan, as shown in Figure 1.5(a). In Figure 1.5(a), m_1 is the equivalent mass of the ith floor, and k_1 is the equivalent stiffness of the ith frame.

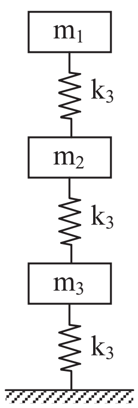

Figure 1.5 (a) Mathematical model.

Problem 1.6

Determine the equivalent stiffness of a linear spring when a linear one-degree-of-freedom spring-mass model is used to replace the system shown in Figure 1.6. Choose x as the generalized coordinate, and neglect the mass of the cantilever beam.

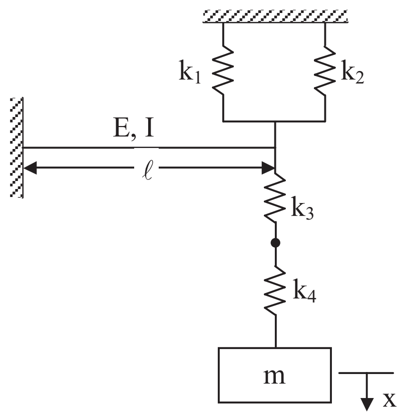

Figure 1.6 Vibrating system.

Solution to Problem 1.6

The cantilever beam behaves in the same way as a linear spring. The equivalent spring stiffness of the two springs k_1 and k_2 is

$$k_{12} = k_1 + k_2 \tag{1.6.1}$$

The displacement of the end of the upper spring k_{12} and the end of the cantilever beam are the same. In other words, the beam is in parallel with the upper springs k_1 and k_2, as shown in Figures 1.6(a) and (b).

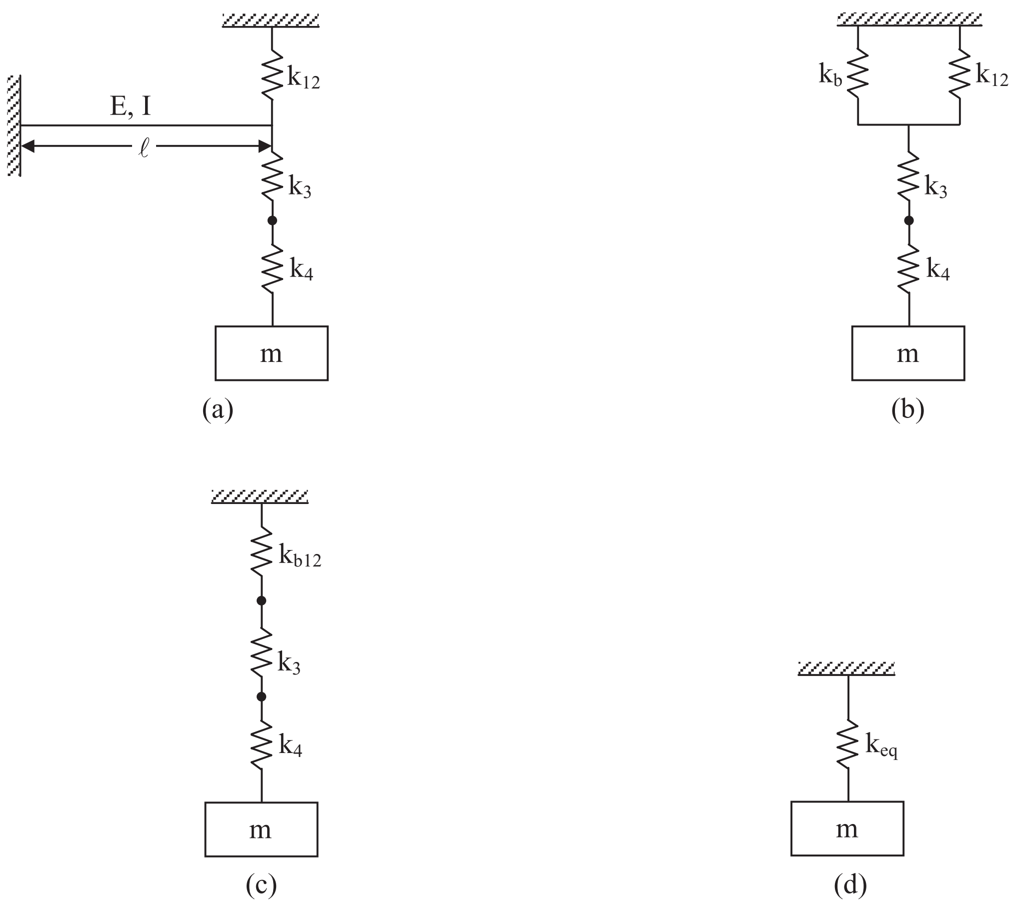

Figure 1.6 (a) An equivalent mathematical model; (b) another equivalent mathematical model; (c) a third equivalent mathematical model; and (d) a fourth equivalent mathematical model.

In Figures 1.6(a) and (b),

$$k_{b12} = k_b + k_{12} \tag{1.6.2}$$

in which k_b is the equivalent stiffness of the cantilever beam at its end

$$k_b = \frac{3EI}{\ell^3} \tag{1.6.3}$$

The total deflection of the system is the deflection of the beam plus the change in length of the two lower springs. Hence, the lower two springs are in series with the beam and upper spring, as shown in Figures 1.6(c) and (d).

Hence,

$$\frac{1}{k_{eq}} = \frac{1}{k_{b12}} + \frac{1}{k_3} + \frac{1}{k_4} = \frac{k_3k_4 + k_{b12}k_4 + k_{b12}k_3}{k_{b12}k_3k_4}$$

or

$$k_{eq} = \frac{k_{b12}\,k_3\,k_4}{k_3k_4 + k_{b12}k_4 + k_{b12}k_3} \tag{1.6.4}$$

where k_{b12}, and k_b are defined in Eqs. 1.6.2 and 1.6.3.

Problem 1.7

Determine the equivalent stiffness of the system shown in Figure 1.7 when a linear one-degree-of-freedom spring-mass model is used to replace the system. Choose x as the generalized coordinates, and neglect the mass of the beam.

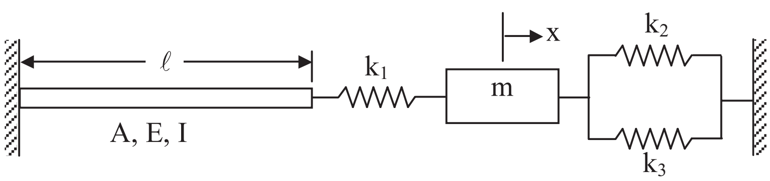

Figure 1.7 Vibrating system.

Solution to Problem 1.7

The equivalent stiffness of the bar is

$$k_b = \frac{AE}{\ell} \tag{1.7.1}$$

The bar acts as a spring in series with the spring of stiffness k_1, as shown in Figure 1.7(a). The springs k_2 and k_3 are in parallel, or

$$k_{23} = k_2 + k_3 \tag{1.7.2}$$

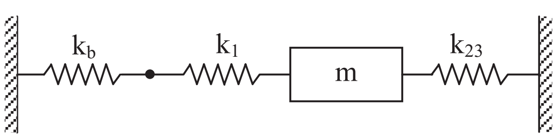

Figure 1.7 (a) Equivalent mathematical model.

The equivalent stiffness of the springs to the left of the block is

$$\frac{1}{k_{b1}} = \frac{1}{k_b} + \frac{1}{k_1} \tag{1.7.3}$$

or

$$\frac{1}{k_{b1}} = \frac{k_b\,k_1}{k_1 + k_b} \tag{1.7.4}$$

Now, the combination to the left of the block is in parallel with the combination to the right of the block (i.e., the same displacements and forces add up).

Hence, the equivalent stiffness of the entire combination is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_{b1}} + \frac{1}{k_{23}} \tag{1.7.5}$$

or

$$k_{eq} = \frac{k_{b1}\,k_{23}}{k_{b1} + k_{23}} \tag{1.7.6}$$

where k_{23} and k_{b1} are defined in Eqs. 1.7.2 and 1.7.4, respectively.

Problem 1.8

Figure 1.8 shows a disk attached to the end of a uniform circular steel beam having three degrees of freedom that are kinetically independent. The three degrees of freedom are the longitudinal displacement, transverse deflection, and angular displacement. The radius of the circular beam is 15 mm, and the length of the beam is 100 cm. Young's modulus of elasticity and the shear modulus of rigidity are 200×10^9 N/m^2 and 100×10^9 N/m^2, respectively. Determine the longitudinal stiffness, torsional stiffness, and transverse stiffness of the beam.

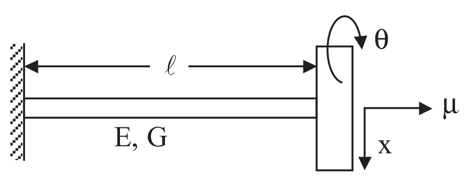

Figure 1.8 Vibrating system.

Solution to Problem 1.8

For the beam, the area of cross section is

$$A = \pi r^2 = \pi\left(0.15\right)^2 = 0.0707 \text{ m}^2$$

$$J = \frac{\pi r^4}{2} = \frac{\pi\left(0.15\right)^4}{2} = 7.952 \times 10^{-4}\text{m}^4$$

$$I = \frac{\pi r^4}{4} = \frac{\pi}{4}\left(0.15\right)^4 = 3.976 \times 10^{-4}\text{m}^4$$

The longitudinal stiffness is

$$k_\ell = \frac{AE}{\ell} = \frac{(0.0707)(200 \times 10^9)}{1} = 14.14 \times 10^9 \text{ N/m}$$

The torsional stiffness is

$$k_t = \frac{JG}{\ell} = \frac{(7.952 \times 10^{-4})(100 \times 10^9)}{1} = 7.952 \times 10^7 \text{ N-m/rad}$$

The transverse stiffness is

$$k_x = \frac{3EI}{\ell^3} = \frac{3(200 \times 10^9)(3.976 \times 10^{-4})}{1} = 2.386 \times 10^8 \text{ N/m}$$

Problem 1.9

Determine the equivalent stiffness of a single spring replacing the one-degree-of-freedom spring-mass system shown in Figure 1.9.

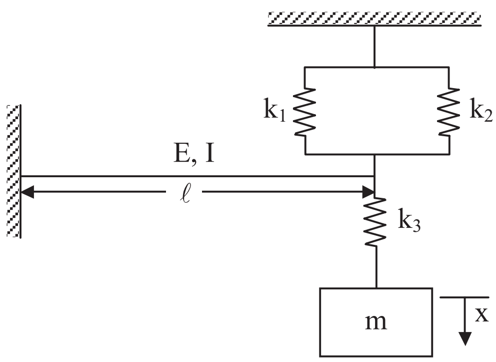

Figure 1.9 Vibrating system.

Solution to Problem 1.9

The cantilever beam behaves in the same way as a linear spring. Note that the displacement of the end of the upper two springs and the end of the cantilever beam are the same. Hence, the beam is parallel with the upper spring, as shown in Figure 1.9(a).

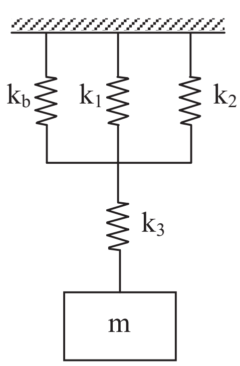

Figure 1.9 (a) Equivalent mathematical model.

In the above figure, the quantity k_b is defined as

$$k_b = \frac{3EI}{\ell^3} \tag{1.9.1}$$

Here, k_b is the equivalent stiffness of the cantilever beam at its end.

Hence, the equivalent stiffness of the beam and the springs k_1 and k_2 in parallel is

$$k_{e1} = \frac{3EI}{\ell^3} + (k_1 + k_2) \tag{1.9.2}$$

The total deflection of the system is the deflection of the beam plus the change in length of the lower spring k_3. Therefore, the lower spring is in series with the beam and upper spring k_{e1}.

Applying the equation for a series configuration gives

$$\frac{1}{k_{eq}} = \left(\frac{1}{k_{e1}} + \frac{1}{k_3}\right) = \left(\frac{k_3 + k_{e1}}{k_{e1}\,k_3}\right) = \frac{k_3 + \left[\frac{3EI}{\ell^3} + (k_1 + k_2)\right]}{\left[\frac{3EI}{\ell^3} + (k_1 + k_2)\right]k_3} \tag{1.9.3}$$

or

$$k_{eq} = \frac{k_3 + \left[\frac{3EI}{\ell^3} + (k_1 + k_2)\right]}{\left[k_1 + k_2 + k_3 + \frac{3EI}{\ell^3}\right]} \tag{1.9.4}$$

Problem 1.10

Determine the equivalent stiffness of the system shown in Figure 1.10. Assume that the cantilever beam is of negligible mass and has a length ℓ.

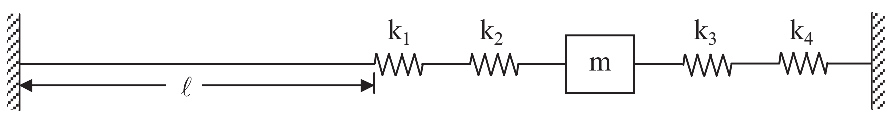

Figure 1.10 Vibrating system.

Solution to Problem 1.10

The equivalent stiffness of the cantilever bar is

$$k_b = \frac{AE}{\ell} \tag{1.10.1}$$

The cantilever bar acts as a spring in series with the springs of stiffness k_1 and k_2, as shown in Figures 1.10(a) and (b), where

$$\frac{1}{k_{e1}} = \frac{1}{k_b} + \frac{1}{k_1} + \frac{1}{k_2} = \left(\frac{k_1k_2 + k_2k_b + k_1k_b}{k_1k_2k_b}\right)$$

or

$$k_{e1} = \frac{k_1 k_2 k_b}{\left(k_1 k_2 + k_2 k_b + k_1 k_b\right)} \tag{1.10.2}$$

and

$$\frac{k}{k_{e2}} = \frac{1}{k_3} + \frac{1}{k_4} = \frac{k_3 + k_4}{k_3 k_4}$$

or

$$k_{e2} = \frac{k_3 k_4}{k_3 + k_4} \tag{1.10.3}$$

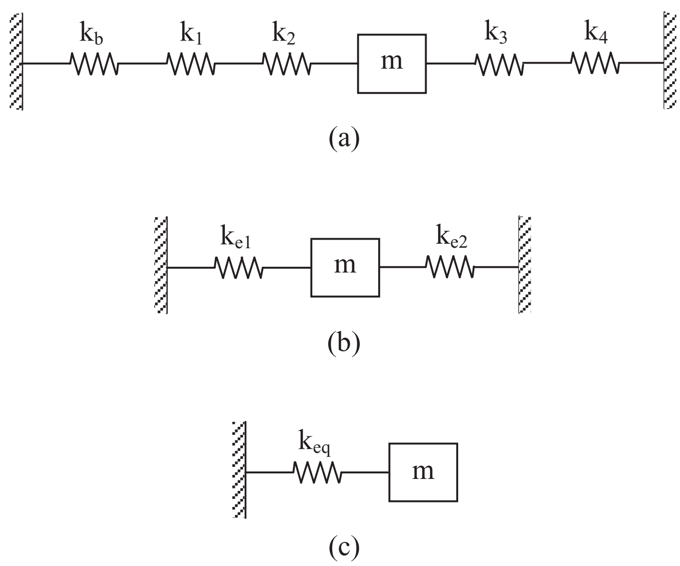

Figure 1.10 (a) An equivalent mathematical model; (b) another equivalent mathematical model; and (c) a third mathematical model.

Now, the combination to the left of the block is in parallel with the combination to the right of the block with the same displacements and forces.

Hence, the equivalent stiffness of the system is

$$k_{eq} = k_{e1} + k_{e2} \tag{1.10.4}$$

where k_{e1} and k_{e2} are defined in Eqs. 1.10.2 and 1.10.3, respectively.

Problem 1.11

Determine the stiffness for the following cases:

a. The torsional stiffness of a solid aluminum shaft of length 2 m and radius 30 cm. The shear modulus of rigidity for aluminum is 41 × 109 N/m^2.

b. The torsional stiffness of a steel annular shaft of length 3 m, inner radius 15 cm, and outer radius 20 cm. The shear modulus of rigidity for steel is 80 × 109 N/m^2.

c. The longitudinal stiffness of a beam of length 1 m, area of cross section of 3×10^{-4} m^2, and Young's modulus of elasticity of 200×109 N/m^2.

d. The transverse stiffness of a steel cantilever beam of length 1 m, moment of inertia of 7×10^{-9} m^4, and Young's modulus of elasticity of 200×109 N/m^2.

Solution to Problem 1.11

a. The polar moment of inertia of the shaft is

$$J = \frac{\pi}{2} r^4 = \frac{\pi}{2} (0.30)^4 = 1.272 \times 10^{-2} \text{ m}^4$$

The torsional stiffness is

$$k_t = \frac{JG}{\ell} = \frac{(1.272 \times 10^{-2})(41 \times 10^9)}{2} = 2.608 \times 10^8 \text{ N-m/rad}$$

b. The polar moment of inertia of the shaft is

$$J = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{2}\left[(0.2)^4 - (0.5)^4\right] = 1.09375 \times 10^{-3} \text{ m}^4$$

The torsional stiffness of the shaft is

$$k_t = \frac{JG}{\ell} = \frac{(1.09375 \times 10^{-3})(80 \times 10^9)}{3} = 2.9167 \times 10^7 \text{ N-m/rad}$$

c. The longitudinal stiffness is

$$k_\ell = \frac{AE}{\ell} = \frac{(3 \times 10^{-4})(200 \times 10^9)}{3} = 6 \times 10^7 \text{ N/m}$$

d. The transverse stiffness is

$$k_y = \frac{3EI}{\ell^3} = \frac{3(200 \times 10^9)(7 \times 10^{-9})}{(1)^3} = 4200 \text{ N/m}$$

Problem 1.12

Determine the equivalent spring stiffness for the system shown in Figure 1.12.

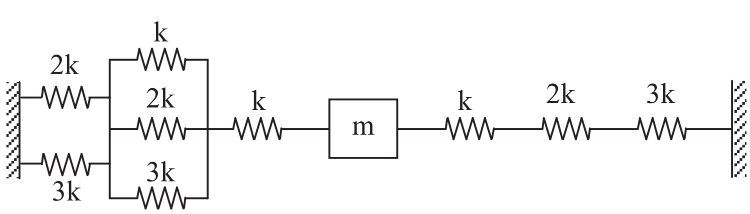

Figure 1.12 Vibrating system.

Solution to Problem 1.12

The two sets of parallel springs are replaced by equivalent springs, stiffnesses of which are shown in Figure 1.12(a).

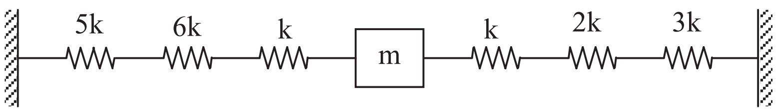

Figure 1.12 (a) Equivalent mathematical model.

Next, the springs to the right of the block are in series, as are springs to the left of the block. Both sets of series springs are replaced by equivalent springs, stiffnesses of which are shown in Figure 1.12(b), where k_1 and k_2 are given by

$$\frac{1}{k_1} = \left(\frac{1}{5k} + \frac{1}{6k} + \frac{1}{k}\right) = \frac{(6+5+30)}{30k} = \frac{41}{30k}$$

or

$$k_1 = \left(\frac{30}{41}\right)k$$

$$\frac{1}{k_2} = \left(\frac{1}{k} + \frac{1}{2k} + \frac{1}{2k}\right) = \frac{(6+3+2)}{6k} = \frac{11}{6k}$$

or

$$k_2 = \frac{6k}{11}$$

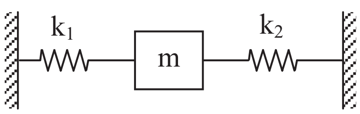

Figure 1.12 (b) Another equivalent mathematical model.

Now, the springs on the left of the block and the springs on the right of the block are in parallel (i.e., have the same displacement and forces). Hence, they are replaced by

$$k_{eq} = k_1 + k_2 = \left(\frac{576k}{451}\right)$$

which is a single spring of equivalent stiffness k_{eq}, as shown in Figure 1.12(c).

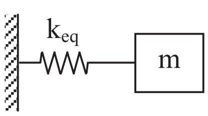

Figure 1.12 (c) Equivalent mathematical model.

Problem 1.13

At $t = 0$, a particle of mass 2 kg has zero velocity, but its speed is increasing at a constant rate of 1 m/s^2. After the particle has traveled 5 m, the local radius of curvature of the path of the particle is 50 m.

a. Determine the speed of the particle after it travels 5 m.

b. Determine the magnitude of the acceleration of the particle after it travels 5 m.

c. Determine the time for the particle to travel 5 m.

Solution to Problem 1.13

Let s(t) be the displacement of the particle, measured from $t = 0$. The velocity is given by

$$v(t) = \int_0^t \frac{dv}{dt}\,dt + v(0) = \int_0^t 1\,dt = 1\,t$$

Because $v = ds/dt$, the displacement of the particle is given by

$$s(t) = \int_0^t v\,dt + s(0) = \int_0^t 1t\,dt = 0.5\,t^2$$

when

$$s = 5\text{ m}$$

$$5\text{ m} = 0.5\,t^2$$

or

$$t = 3.162\text{ s}$$

a. The velocity when $s = 5$ m is

$$v = (1)(3.162) = 3.162\text{ m/s}^2$$

b. As the particle is traveling along a curved path, its acceleration has two components: a tangential component equal to the rate of change of the velocity

$$a_t = \frac{dv}{dt} = 1\text{ m/s}^2$$

and a normal component directed toward the center of curvature.

$$a_n = \frac{v^2}{r} = \frac{(3.162\text{ m/s}^2)}{50\text{ m}} = 0.2\text{ m/s}^2$$

The magnitude of acceleration at this instant is

$$|a| = \sqrt{a_1^2 + a_2^2} = \sqrt{(1\text{ m/s}^2)^2 + (0.2\text{ m/s}^2)^2}$$

$$|a| = 1.0198\text{ m/s}^2$$

c. The time for the particle to travel 5 m is calculated as $t = 3.162$ s.

Problem 1.14

Obtain the differential equations of motion for the two-degrees-of-freedom vibrating system shown in Figure 1.14.

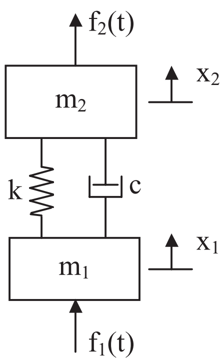

Figure 1.14 Two-degrees-of-freedom system.

Solution to Problem 1.14

Assuming

$$x_1 > x_2 > 0 \qquad \text{(spring in compression)} \tag{1.14.1}$$

This implies

$$\dot{x}_1 > \dot{x}_2 > 0 \tag{1.14.2}$$

The free body diagram of the system is shown in Figure 1.16(a). Applying Newton's second law of motion for the masses m_1 and m_2, we get

$$\sum F_x = ma_{cx} \tag{1.14.3}$$

$$f_1(t) - c(\dot{x}_1 - \dot{x}_2) - k(x_1 - x_2) = m_1\ddot{x}_1 \tag{1.14.4}$$

$$f_2(t) - c(\dot{x}_1 - \dot{x}_2) - k(x_1 - x_2) = m_2\ddot{x}_2 \tag{1.14.5}$$

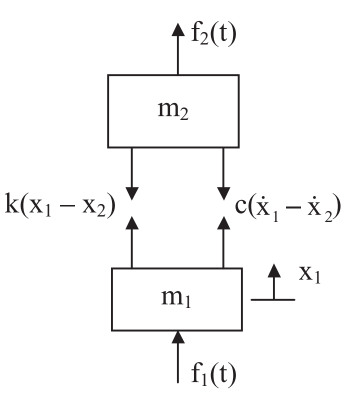

Figure 1.14 (a) Equivalent mathematical model.

Equations 1.14.1 and 1.14.5 can be written in the standard input-output form as

$$m_1\ddot{x}_1 - c\dot{x}_1 - c\dot{x}_2 + kx_1 - kx_2 = f_1(t) \qquad (1.14.6)$$

$$m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + kx_1 - kx_2 = f_1(t) \qquad (1.14.6)$$

$$m_2\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 = f_2(t) \qquad (1.14.7)$$

Problem 1.15

For the translational system shown in Figure 1.15, derive the differential equations of motion.

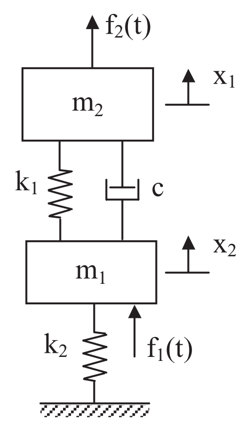

Figure 1.15 Torsional vibrating system.

Solution to Problem 1.15

Assuming the springs are in tension, we have

$$x_1 > x_2 > 0 \qquad (1.15.1)$$

Hence,

$$\dot{x}_1 > \dot{x}_2 > 0 \qquad (1.15.2)$$

The free body diagram is shown in Figure 1.15(a).

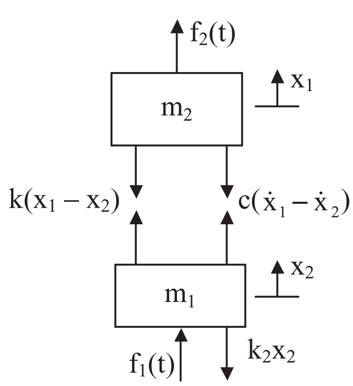

Figure 1.15 (a) Equivalent mathematical model.

Applying Newton's second law for the masses m_1 and m_2, we have

$$\sum F_x = ma_{cx} \tag{1.15.3}$$

$$f_1(t) - c(\dot{x}_1 - \dot{x}_2) - k(x_1 - x_2) = m_1\ddot{x}_1 \tag{1.15.4}$$

$$f_2 t - k_2 x_2 + c(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = m_2\ddot{x}_2 \tag{1.15.5}$$

Equations 1.15.4 and 1.15.5 can be written in the standard input-output form as

$$m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + k_1x_1 - k_2x_2 = f_1(t) \tag{1.15.6}$$

$$m_2\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - k_2x_1 + (k_1 + k_2)x_2 = f_2(t) \tag{1.15.7}$$

or in the standard matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{1.15.8}$$

Problem 1.16

A circular disk of mass m and radius r rolls, without slip, on a horizontal plane along a straight line. Derive the differential equation of motion for the rotational system.

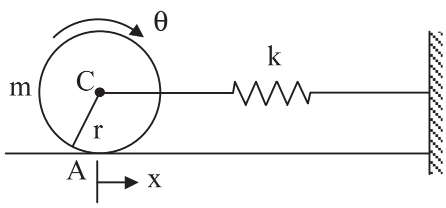

Figure 1.16 Single-degree-of-freedom system.

Solution to Problem 1.16

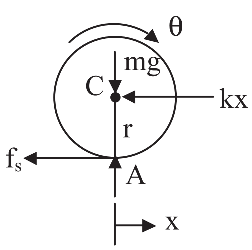

Figure 1.16 (a) Equivalent mathematical model.

The free body diagram is shown in Figure 1.16(a). The instantaneous center is the point A.

Applying

$$\sum M_A = I_A \alpha \qquad (1.16.1)$$

where

$$I_A = I_C + md^2 \qquad (1.16.2)$$

Equation 1.16.1 can be written as

$$r(kx) = I_A \ddot{\theta} \qquad (1.16.3)$$

For the no-slip constraint,

$$x = r\theta \qquad (1.16.4)$$

Equation 1.16.3 becomes

$$I_A \ddot{\theta} = kr^2\theta = 0 \qquad (1.16.5)$$

where

$$I_A = I_C + md^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

Problem 1.17

For the torsional system shown in Figure 1.17, derive the differential equations of motion. The mass moments of inertia of disks 1 and 2 are J_1 and J_2, and the torsional stiffnesses are k_1 and k_2, respectively.

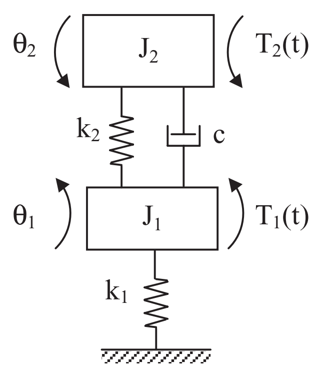

Figure 1.17 Torsional vibrating system.

Solution to Problem 1.17

The free body diagram for $\theta_1 > \theta_2 > 0$ is shown in Figure 1.17(a).

Applying the moment equation about the mass centers, along the rotation axis, we can write

$$\sum M_C = I_C \alpha \qquad (1.17.1)$$

For J_1,

$$T_1(t) - k_1\theta - k_2(\theta_1 - \theta_2) - c(\theta_1 - \theta_2) = J_1\ddot{\theta}_1 \qquad (1.17.2)$$

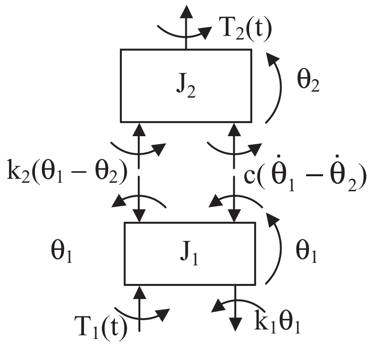

Figure 1.17 (a) Free body diagram.

For J_2,

$$T_2(t) + k_2(\theta_1 - \theta_2) + c(\theta_1 - \theta_2) = J_2\ddot{\theta}_2 \tag{1.17.3}$$

Equations 1.17.2 and 1.17.3 can be rearranged as

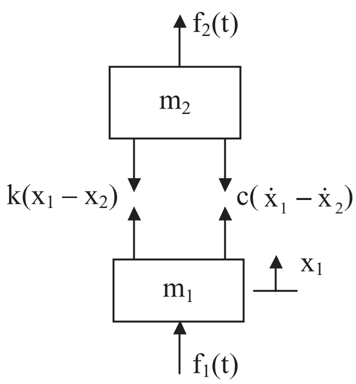

Figure 1.17 (b) Free body diagram.

$$J_1\ddot{\theta}_1 + c\dot{\theta}_1 - c\dot{\theta}_2 + (k_1 + k_2)\theta_1 - k_2\theta_2 = T_1(t) \tag{1.17.4}$$

$$J_2\ddot{\theta}_2 - c\dot{\theta}_1 + c\dot{\theta}_2 - k_2\theta_1 + k_2\theta_2 = T_2(t) \tag{1.17.5}$$

Equations 1.17.4 and 1.17.5 can be written in matrix form as

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} \tag{1.17.6}$$

Problem 1.18

For the mixed two-degrees-of-freedom mechanical pendulum system shown in Figure 1.18, derive the differential equations of motion. Attached to the cart of mass m, which moves in a horizontal direction, is a pendulum that consists of a mass-less rod of length ℓ and a concentrated mass of m_1 at the rod tip.

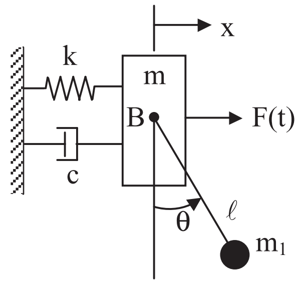

Figure 1.18 Two-degrees-of-freedom mechanical pendulum system.

Solution to Problem 1.18

The free body diagram is shown in Figures 1.18(a) and (b).

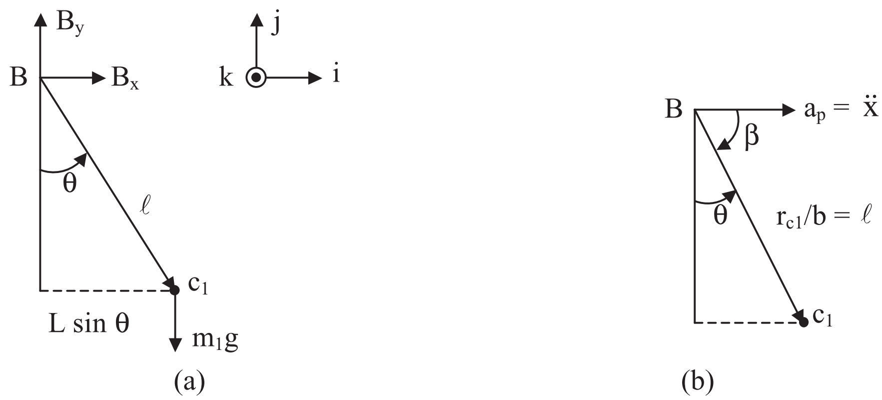

Figure 1.18 Free-body diagrams: (a) forces, and (b) position and acceleration vectors.

The moment equation is

$$\sum \mathbf{M}_p = I_b\alpha + m\,\mathbf{r}_{c/b} \times a_b \tag{1.18.1}$$

Now,

$$\text{rcb} \times \text{ab} = |r_{c/b}||a_b|\sin\beta = \ell\ddot{x}\sin\left(\frac{\pi}{2} - \theta\right) = \ell\ddot{x}\cos\theta \tag{1.18.2}$$

Because

$$I_b = m\ell^2$$

$$-\ell\sin\theta\, m_1 g = m_1\ell^{2\theta} + m\ell\ddot{x}\cos\theta \tag{1.18.3}$$

The free body diagram for the force equation is shown in Figures 1.18(c) and (d).

Applying Newton's second law for the system of rod and cart in the x direction gives

$$\sum_{j=1}^{3} F_{jx} = m_1 a_{cx} = \sum_{i=1}^{2} m_i a_{cix} = m_1 a_{c1x} + m_2 a_{c2x} \tag{1.18.4}$$

Because

$$a_{ci} = a_p + a_{ci/b} \tag{1.18.5}$$

$$\mathbf{a}_{cix} = \mathbf{a}_{bx} + \left(\mathbf{a}_{ci/b}\right)x \tag{1.18.6}$$

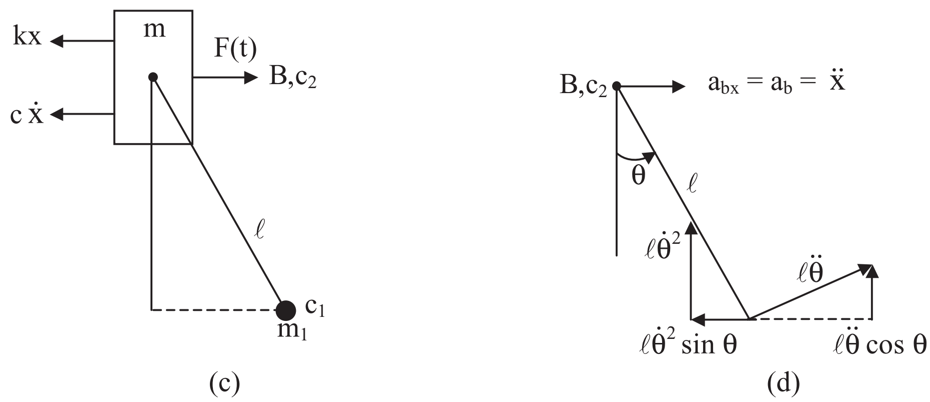

Figure 1.18 Free body diagrams: (c) horizontal force components, and (d) position and acceleration vectors.

Therefore, Eq. 1.18.4 can be written as

$$F(t) - kx - c\dot{x} = m_1\left(\ddot{x} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta\right) + m\ddot{x} \tag{1.18.7}$$

Hence, the differential equations of motion, Eqs. 1.18.3 and 1.18.7, can be written as

$$m_1\ell^{2\theta} + m_1\ell\ddot{x}\cos\theta + m_1 g\ell\sin\theta = 0 \tag{1.18.8}$$

and

$$\left(m + m_1\right)\ddot{x} + m_1\left(\ell\ddot{\theta}\cos\theta = \ell\theta^2\sin\theta\right) + c\dot{x} + kx = F(t) \tag{1.18.9}$$

For small angle motions,

$$\theta << 1$$
$$\cos\theta = 1$$
$$\sin\theta = \theta$$
$$\theta^2 = 0$$

Equations 1.18.8 and 1.18.9 become

$$m_1\ell^{2\theta} + m_1\ell\ddot{x} + m_1 g\ell\ \theta = 0 \tag{1.18.10}$$

$$\left(m + m_1\right)\ddot{x} + m_1\ell\ddot{\theta} + c\dot{x} + kx = F(t) \tag{1.18.11}$$

or in matrix form,

$$\begin{bmatrix} m_1\ell^2 & m\ell \\ m\ell & m + m_1 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} m_1 g\ell & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{1.18.12}$$

Problem 1.19

Figure 1.19 shows an inverted pendulum system that consists of a cart of mass m and a uniform rod of length ℓ and mass m_1. A concentrated mass m_3 is attached at the tip of the rod as shown. Assuming the rod is pivoted to the cart, derive the differential equations of motion for this system. The system is constrained to move in a vertical plane, and the cart rolls without slipping in a horizontal direction.

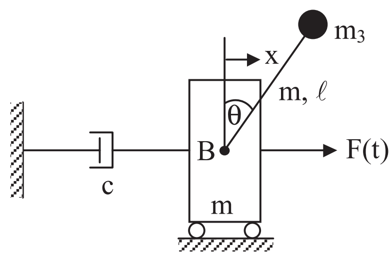

Figure 1.19 Inverted pendulum system.

Solution to Problem 1.19

The free body diagrams for obtaining the moment equations are shown in Figures 1.19(a) and 1.19(b).

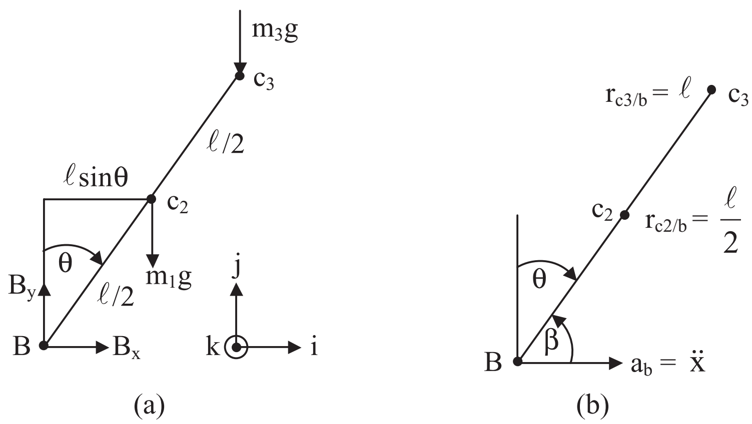

Figure 1.19 Free body diagrams: (a) forces, and (b) position and acceleration vectors.

The moment equation can be written as

$$\sum \mathbf{M}_p = I_b\alpha + m_1\mathbf{r}_{c/b} \times \mathbf{a}_b \tag{1.19.1}$$

or

$$\sum_{j=1}^{N} \mathbf{M}_{bj} = \left(\sum_{i=1}^{n} I_{bi}\right)\alpha + \sum_{i=1}^{n}(m_i\mathbf{r}_{ci/b} \times \mathbf{a}_b) \tag{1.19.2}$$

For n = 2 and N = 2, the moment equation is

$$\sum_{j=1}^{N=2} \mathbf{M}_{bj} = \left(\sum_{i=1}^{n=2} I_{bi}\right)\alpha + \sum_{i=1}^{n=2}\left(m_i\mathbf{r}_{ci/b} \times \mathbf{a}_b\right) \tag{1.19.3}$$

Now,

$$\mathbf{r}_{ci/b} \times \mathbf{a}_b = |\mathbf{r}_{ci/b}||\mathbf{a}_b|\sin\beta = |\mathbf{r}_{ci/b}|\ddot{x}\cos\theta \tag{1.19.4}$$

Hence,

$$\frac{\ell}{2}\sin\theta\, m_1 g + \ell\sin\theta\, m_3 g = I_b\ddot{\theta} + m_1\frac{\ell}{2}\ddot{x}\cos\theta + m_3\ell\ddot{x}\cos\theta \tag{1.19.5}$$

where

$$I_b = \sum_{i/1}^{n=1} I_{ci} m_i d_i^2 = I_{c1} + m_i d_1^2 + I_{c2} + m_2 d_2^2 \tag{1.19.6}$$

$$I_b = \left[\frac{1}{12} m_1 \ell^2 + m_1 \left(\frac{\ell}{2}\right)^2\right] + \left[0 + m_3 \ell^2\right] = \frac{1}{3} m_1 \ell^2 + m_3 \ell^2 = \left(\frac{m_1}{3} + m_3\right)\ell^2 \tag{1.19.7}$$

The free body diagrams for obtaining the translation equation are shown in Figures 1.19(c) and 1.19(d).

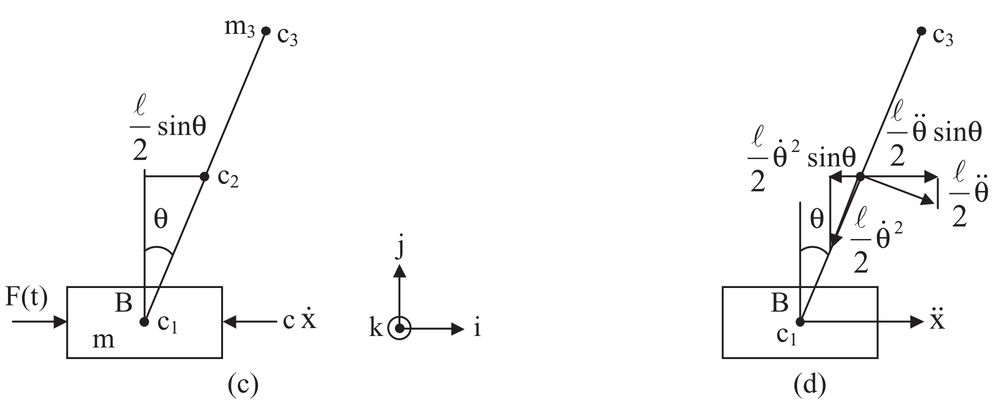

Figure 1.19 Free body diagram: (c) horizontal force components, and (d) position and acceleration vectors.

Applying Newton's second law for the system of cart and rod in the x direction, we have

$$\sum_{j=1}^{N=2} F_{jx} = ma_{cx} = \sum_{i=1}^{n=3} m_i a_{cix} = m_1 a_{c1x} = m_2 a_{c2x} = m_3 a_{c3x} \tag{1.19.8}$$

Now,

$$\mathbf{a}_{ci} = \mathbf{a}_p + \mathbf{a}_{ci/b} \tag{1.19.9}$$

$$a_{cix} = a_{bx} + \left(a_{ci/b}\right)x \tag{1.19.10}$$

Hence,

$$\begin{aligned} F(t) - c\dot{x} = m\ddot{x} &+ m_1\left(\ddot{x} + \frac{\ell}{2}\ddot{\theta}\cos\theta - \frac{\ell}{2}\dot{\theta}^2 \sin\theta\right) \\ &+ m_3\left(\ddot{x} + \ell\ddot{\theta}\cos\theta - \ell\theta^2 \sin\theta\right) \end{aligned} \tag{1.19.11}$$

Equations (1.19.5) and (1.19.11) can be rearranged as

$$I_b\ddot{\theta} + \left(\frac{m_1}{2} + m_3\right)\ell\ddot{x}\cos\theta - \left(\frac{m_1}{2} + m_3\right)g\ell\sin\theta = 0 \tag{1.19.12}$$

and

$$\left(m + m1 + m3\right)\ddot{x} + \left(\frac{m_1}{2} + m_3\right)\ell\ddot{\theta}\cos\theta - \left(\frac{m_1}{2} + m_3\right)\ell\dot{\theta}^2 \sin\theta + c\dot{x} = F(t) \tag{1.19.13}$$

or Eqs. 1.19.12 and 1.19.13 can be written as

$$I_b\ddot{\theta} + M_1\ell\ddot{\theta}\cos\theta - M_1 g\ell\sin\theta = 0 \tag{1.19.14}$$

and

$$M_{2\ddot{x}} + M_1\ell\ddot{\theta}\cos\theta - M_1\ell\dot{\theta}^2\sin\theta + c\dot{x} = F(t) \tag{1.19.15}$$

For small angle motions,

$$\theta << 1$$
$$\cos\theta = 1$$
$$\sin\theta = \theta$$
$$\theta^2 = 0$$

Equations 1.19.14 and 1.19.15 become

$$I_b\ddot{\theta} + M_1\ell\ddot{x} - M_1 g\ell\theta = 0 \tag{1.19.16}$$

and

$$M_1\ell\ddot{\theta} + M_{2\ddot{x}} + c\dot{x} = F(t) \tag{1.19.17}$$

where

$$M_1 = \frac{m_1}{2} + m_3$$

and

$$M_2 = m + m_1 + m_3$$

Equations 1.19.16 and 1.19.17 can be written in second-order matrix form as

$$\begin{bmatrix} I_b & M_1\ell \\ M_1\ell & M_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} -M_1 g\ell & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{1.19.18}$$

Problem 1.20

Derive the differential equations of motion for the three-degrees-of-freedom gear-train system shown in Figure 1.20. The moments of inertia of the gears are J_1, J_2, J_3, and J_4, as shown. The radii of the gears 2 and 3 are r_2 and r_3, respectively. The applied torques on gears 1 and 4 are $T_1(t)$ and $T_4(t)$, respectively. Assume the gears are rigid and have no backlash.

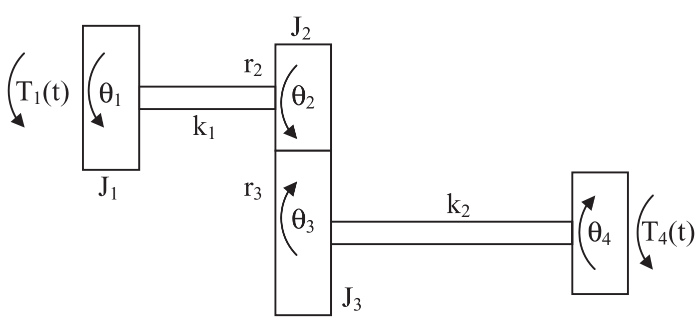

Figure 1.20 Three-degrees-of-freedom system.

Solution to Problem 1.20

Assume the shafts are long and hence flexible. The free body diagrams of the gears are shown in Figure 1.20(a). Because the gears are assumed to be rigid and have no backlash, the geometric constraint is given by

$$r_2\theta_2 = r_2\theta_3 \tag{1.20.1}$$

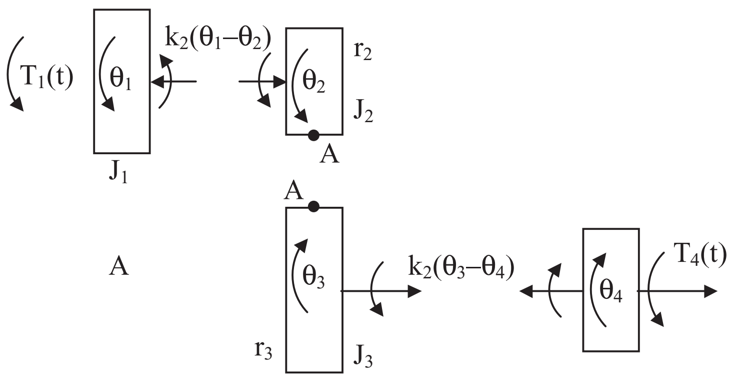

Figure 1.20 (a) Free body diagram.

From Eq. 1.20.1,

$$\theta = \frac{r_2}{r_3}\theta_2$$

and

$$\ddot{\theta}_3 = \frac{r_2}{r_3}\ddot{\theta}_2 \tag{1.20.2}$$

Applying the moment equation about the mass center C,

$$\sum \mathbf{M}_C = I_C\alpha \tag{1.20.3}$$

Gear 1:

$$T_1(t) - k_1(\theta_1 - \theta_2) = J_1\ddot{\theta}_1 \tag{1.20.4}$$

Gear 2:

$$k_1(\theta_1 - \theta_2) - r_2A = J_2\ddot{\theta}_2 \tag{1.20.5}$$

Gear 3:

$$r_2A - k_2(\theta_3 - \theta_4) = J_3\ddot{\theta}_3 \tag{1.20.6}$$

Gear 4:

$$-T_4(t) + k_2(\theta_3 - \theta_4) = J_4\ddot{\theta}_4 \tag{1.20.7}$$

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Substituting Eq. 1.20.2 into Eqs. 1.20.6 and 1.20.7, we get

$$r_3 A - k_2\left(\frac{r_2}{r_3}\theta_2 - \theta_4\right) = J_3\frac{r_2}{r_3}\ddot{\theta}_2 \tag{1.20.8}$$

$$-T_4(t) + k_3\left(\frac{r_2}{r_3}\theta_2 - \theta_4\right) = J_4\ddot{\theta}_4 \tag{1.20.9}$$

From Eqs. 1.20.8 and 1.20.9, we have

$$k_1(\theta_1 - \theta_2) - \frac{r_2}{r_3}k_2\left(\frac{r_2}{r_3}\theta_2 - \theta_4\right) = \left[J_2 + J_3\left(\frac{r_2}{r_3}\right)^2\right]\ddot{\theta}_2 \tag{1.20.10}$$

Hence, the equations of motion are

$$J_1\ddot{\theta}_1 + k_1\theta_1 - k_2\theta_2 = T_1(t) \tag{1.20.11}$$

$$\left[J_2 + J_3\left(\frac{r_2}{r_3}\right)^2\right]\ddot{\theta}_2 - k_1\theta_1 + \left[k_1 + k_2\left(\frac{r_2}{r_3}\right)^2\right]\theta_2 - \frac{r_2}{r_3}k_2\theta_4 = 0 \tag{1.20.12}$$

$$J_4\ddot{\theta}_4 - k_2\frac{r_2}{r_3}\theta_2 + k_2\theta_4 = -T_4(t) \tag{1.20.13}$$

or in matrix form,

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 + J_3\left(\frac{r_2}{r_3}\right)^2 & 0 \\ 0 & 0 & J_4 \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3\end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2\left(\frac{r_2}{r_3}\right)^2 & -\left(\frac{r_2}{r_3}\right)k_2 \\ 0 & -\left(\frac{r_2}{r_3}\right)k_2 & k_2 \end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2 \\ \theta_3\end{bmatrix} = \begin{bmatrix} T_1(t) \\ 0 \\ -T_4(t)\end{bmatrix} \tag{1.20.14}$$

Problem 1.21

Determine the static equilibrium position of a double pendulum shown in Figure 1.21 when a horizontal force F is applied to m_2. Use the generalized coordinates θ_1 and θ_2 and the principle of virtual work.

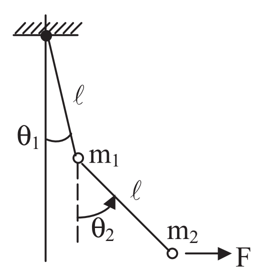

Figure 1.21 Double pendulum.

Solution to Problem 1.21

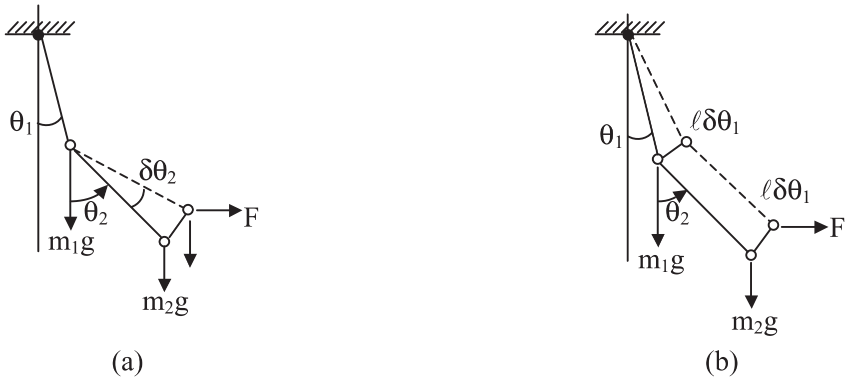

Figure 1.21 Free body diagrams: (a) equilibrium position, and (b) equilibrium position with $\delta\theta_1 = 0$.

When the system is in equilibrium position, a virtual displacement $\delta\theta_2$ is given to θ_2, as shown in Figure 1.21(a).

The equation for the virtual work δw of all the applied forces then is given by

$$\delta w = \left(m_2 g \sin\theta_2\right)\ell\delta\theta_2 + \left(F\cos\theta_2\right)\ell\delta\theta_2 = 0 \tag{1.21.1}$$

Similarly, by giving θ_1 a virtual displacement $\delta\theta_1$ with the equilibrium position (with $\delta\theta_2 = 0$) as shown in Figure 1.21(b), we can write the equation for virtual work δw as

$$\delta w = -\left(m_1 g \sin\theta_1\right)\ell\delta\theta_1 - \left(m_2 g \sin\theta_1\right)\ell\delta\theta_1 + \left(F\cos\theta_1\right)\ell\delta\theta_1 = 0 \tag{1.21.2}$$

Solving Eqs. 1.21.1 and 1.21.2 for the two equilibrium angles gives

$$\tan\theta_1 = \frac{F}{\left(m_1 + m_2\right)g} \tag{1.21.3}$$

and

$$\tan\theta_2 = \frac{F}{m_2 g} \tag{1.21.4}$$

Problem 1.22

A particle of mass m is suspended by a mass-less wire of length

$$r = a + b\cos\omega t \qquad \left(a > b > 0\right)$$

to form a pendulum, whose motion is confined to a plane, as shown in Figure 1.22. Find the equation of motion using D'Alembert's principle.

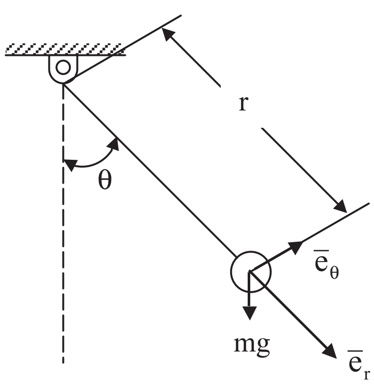

Figure 1.22 Simple pendulum.

Solution to Problem 1.22

Let r and θ be the chosen generalized coordinates. The general expression for the acceleration of the mass m would be

$$\ddot{\bar{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\bar{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\bar{e}_\theta$$

The virtual displacement consistent with the constraint is

$$\delta\bar{r} = r\delta\theta\bar{e}_\theta$$

The applied gravitational force is

$$\bar{F} = mg\cos\theta\bar{e}_r - mg\sin\theta\bar{e}_\theta$$

The equation of motion is obtained as

$$\left(\bar{F} - m\ddot{\bar{r}}\right)\delta\bar{r} = 0$$

$$\left\{mg\cos\theta\bar{e}_r - mg\sin\theta\bar{e}_\theta - m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\bar{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\bar{e}_\theta\right]\right\}\bar{e}_\theta r\delta\theta = 0$$

that is,

$$mg\sin\theta - m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\sin\theta$$

$$\left(a + b\cos\omega t\right)\ddot{\theta} - 2b\omega\dot{\theta}\sin\omega t = -g\sin\theta$$

Problem 1.23

A simple pendulum of length ℓ and mass m is pivoted to the mass M that slides without friction on a horizontal plane, as shown in Figure 1.23. Use Lagrange's equation to determine the equations of motion for the system.

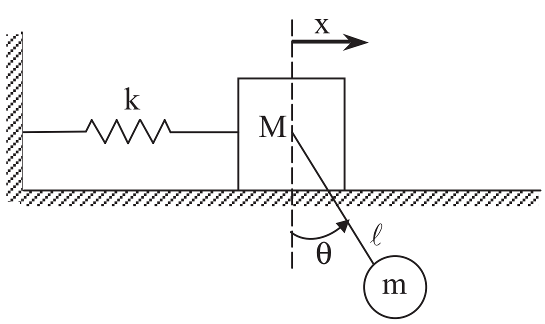

Figure 1.23 Pendulum system.

Solution to Problem 1.23

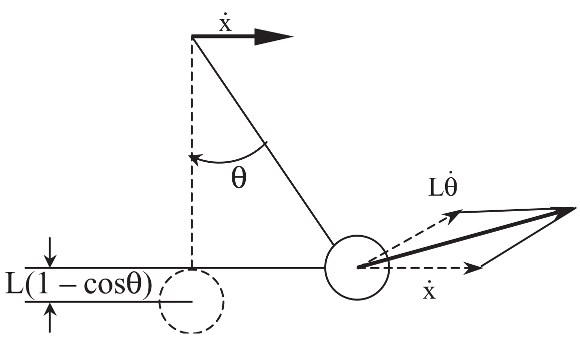

Figure 1.23 Motion of the pendulum system.

Let $x(t)$ denote the displacement of the mass M, and let $\theta(t)$ denote the angular swing of the pendulum. The kinetic energy of the system is due to the motion of mass M and the swing of the pendulum having mass m. The potential energy is derived from the spring (stretch or compress) and the position of the job.

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + L2\dot{\theta}^2 + 2L\dot{x}\,\dot{\theta}\cos\theta\right)$$

The potential energy of the system is

$$\frac{d}{dt}\left[\frac{\partial\left(\text{kinetic energy}\right)}{\partial\dot{x}}\right] = \left(M + m\right) + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\cos\theta$$

$$\frac{\partial\left(\text{kinetic energy}\right)}{\partial x} = 0$$

and

$$\frac{\partial\left(\text{potential energy}\right)}{\partial x} = kx$$

Applying Lagrange's equations, we obtain

$$\left(M + m\right)\ddot{x} + mL\,\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta + kx = 0$$

For small angles of oscillation,

$$\sin\theta = \theta$$

and

$$\cos\theta = 1$$

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Then, neglecting higher-order terms, the equation of motion becomes

$$(M + m)\ddot{x} + mL\,\ddot{\theta} + kx = 0$$

Similarly,

$$\frac{d}{dt}\left[\frac{\partial(\text{kinetic energy})}{\partial\dot{\theta}}\right] = mL2\ddot{\theta} + mL\ddot{x}$$

and

$$\frac{\partial(\text{potential energy})}{\partial\theta} = mLg\theta$$

$$\frac{\partial(\text{kinetic energy})}{\partial\theta} = 0$$

Hence,

$$L\ddot{\theta} + g\theta\ddot{x} = 0$$

Problem 1.24

Figure 1.24 shows a solid cylinder of mass M and radius R rolling without slipping on a horizontal surface. A bob of mass m is pinned to the axis of the cylinder by an arm of length ℓ. Determine the following:

a. The equations of motion for this system using Lagrange's equation

b. The natural frequency of free vibration of the bob

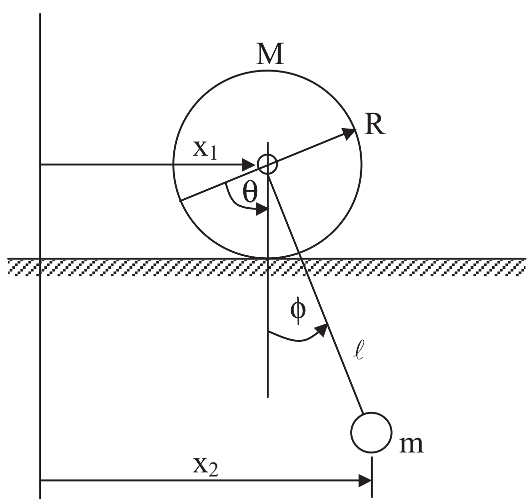

Figure 1.24 Solid cylinder.

Solution to Problem 1.24

a. Taking x_1 and x_2 as the generalized coordinates, the kinetic and potential energies of the system for small values of θ are

$$T = \frac{1}{2}M\dot{x}_1^2 + \left(\frac{1}{2}MR^2\right)\dot{\theta}^2 + \frac{1}{2}m\dot{x}_2^2$$
$$= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}\left(\frac{1}{2}M\dot{x}_1^2\right) + \frac{1}{2}m\dot{x}_2^2 \qquad (1.24.1)$$

$$V = mg\ell(1 - \cos\phi) = (mg\ell/2)\phi^2 = (mg\ell/2)(x_2 - x_1)^2$$

Apply Lagrange's equation with $q_i = x_1$:

$$(d/dt)(\partial T/\partial \dot{x}_1) = M\ddot{x}_1 + \frac{1}{2}M\ddot{x}_1 \qquad (1.24.1)$$

$$\partial V/\partial x_1 = (mg/2\ell)(-2x_2 + 2x_1) \qquad (1.24.2)$$

Therefore,

$$\frac{3}{2}M\ddot{x}_1 + (mg/\ell)(x_1 - x_2) = 0 \qquad (1.24.3)$$

Apply Lagrange's equation with $q_i = x_2$:

$$(d/dt)(\partial T/\partial \dot{x}_2) = M\ddot{x}_2$$

$$\partial V/\partial x_2 = (mg/2\ell)(2x_2 - 2x_1)$$

Therefore,

$$M\ddot{x}_2 + (mg/\ell)(x_2 - x_1) = 0 \qquad (1.24.4)$$

These two equations of motion, Eqs. 1.24.3 and 1.24.4, can be solved by assuming that $x_1 = X\sin\omega t$ and $x_2 = X_2 \sin\omega t$. The results are

$$X_1(mg/\ell) - (3M/2) + X_2(-mg/\ell) = 0 \qquad (1.24.5)$$

and

$$X_1(mg/\ell) + X_2((mg/\ell) - M\omega^2) = 0 \qquad (1.24.6)$$

b. The frequency equation is obtained as

$$(3M/2)\omega^4 - (g/\ell)\omega^2(m + (3M/2)) = 0 \qquad (1.24.7)$$

Solving Eq. 1.24.7, we obtain either

$$\omega = 0$$

or

$$\omega = \sqrt{((1 + 2m/3M)g/\ell)} \text{ rad/s}$$

and

$$X_1/X_2 = 2m/3M$$

Problem 1.25

Derive the differential equations governing the motion of the system shown in Figure 1.25 using Lagrange's equation. Take θ_1 and θ_2 as generalized coordinates; the mass of the slender bar is m.

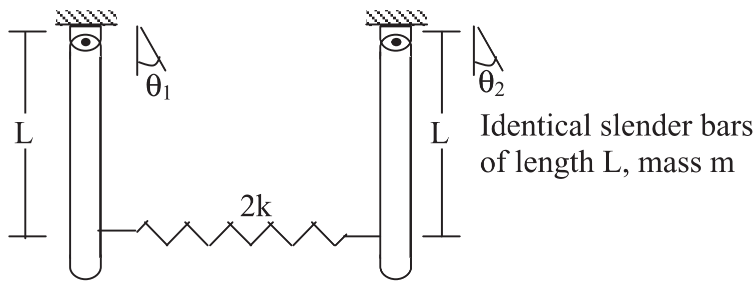

Figure 1.25 System of bars.

Solution to Problem 1.25

The kinetic energy of the system is

$$T = \frac{1}{2}m\left(\frac{1}{2}L\dot{\theta}_1\right)^2 + \frac{1}{12}mL^2\dot{\theta}_1^2 + \frac{1}{2}m\left(\frac{1}{2}L\dot{\theta}_2\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}_2^2$$

The potential energy of the system is

$$V = mg\frac{L}{2}\cos\theta_1 - mg\frac{L}{2}\cos\theta_2\left(2k\right)\left(a\sin\theta_2 - a\sin\theta_1\right)$$

The Lagrangian equation is

$$\begin{aligned} L = {} & \frac{1}{2}\frac{1}{3}mL^2\theta_1^2 + \frac{1}{2}\frac{1}{3}mL^2\theta_2^2 + mg\frac{L}{2}\cos\theta_1 \\ & + mg\frac{L}{2}\cos\theta_2 - \frac{2k}{2}\left(a\sin\theta_2 - a\sin\theta_1\right) \end{aligned}$$

Application of Lagrange's equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{1}{3}mL^2\dot{\theta}_1\right)\left[mg\frac{L}{2}\sin\theta_1 + 2k\left(a\sin\theta_2 - a\sin\theta_1\right)\left(a\cos\theta_1\right)\right] = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{dt}\left(\frac{1}{3}mL^2\dot{\theta}_2\right) + \left[mg\frac{L}{2}\sin\theta_2 + 2k\left(a\sin\theta_2 - a\sin\theta_1\right)\left(a\cos\theta_2\right)\right] = 0$$

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Linearizing and rearranging the resultant equations, we obtain

$$\frac{1}{3}mL^2\ddot{\theta}_1 + \left(mg\frac{L}{2} + 2ka^2\right)\theta_1 - 2ka^2\theta_2 = 0$$

Problem 1.26

Derive the differential equations governing the motion of the system shown in Figure 1.26 using Lagrange's equation and the generalized coordinates x_1, x_2, and x_3.

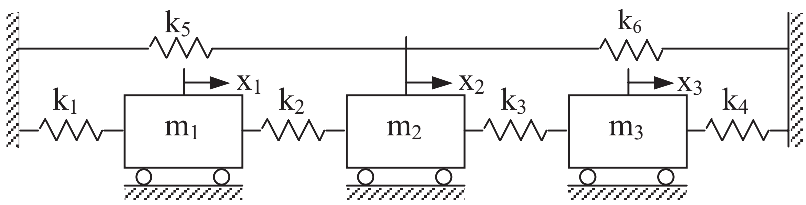

Figure 1.26

Solution to Problem 1.26

Kinetic energy:

Potential energy:

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - x_1\right)^2 + \frac{1}{2}k_3\left(x_3 - x_2\right)^2 + \frac{1}{2}k_4x_3^2 + \frac{1}{2}\left(x_5 + x_6\right)x_2^2$$

$$\frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1} = k_1x_1 + k_2\left(x_2 - x_1\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = 0$$

gives

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2x_2 = 0$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$$

$$\frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \dot{x}_2 = m_2 \ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_2} = k_2\left(x_2 - x_1\right) + k_3\left(x_3 - x_2\right) + \left(x_5 - x_6\right)x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = 0$$

gives

$$m_2\ddot{x}_2 - k_2x_1 + \left(k_2 + k_3 + k_5 + k_6\right)x_2 - k_3x_3 = 0$$

$$\frac{\partial T}{\partial \dot{x}_3} = m_3\dot{x}_3 \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) m3\ddot{x}_3$$

$$\frac{\partial T}{\partial x_3} = 0$$

$$\frac{\partial V}{\partial x_3} = k_3\left(x_3 - x_2\right) + k_4x_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) - \frac{\partial T}{\partial x_3} + \frac{\partial V}{\partial x_3} = 0$$

gives

$$m_3\ddot{x}_3 - k_3x_2 + \left(k_3 + k_4\right)x_3 = 0$$

Problem 1.27

Figure 1.27 shows a three-degrees-of-freedom system. Determine the generalized forces associated with the generalized coordinates $q_1 = \theta_1$, $q_2 = \theta_2$, and $q_3 = \theta_3$, where **F** and **M** are defined as

$$F = \left[F_1 \quad F_2\right]^T$$

$$M = \left[M_1 \quad M_2 \quad M_3\right]^T$$

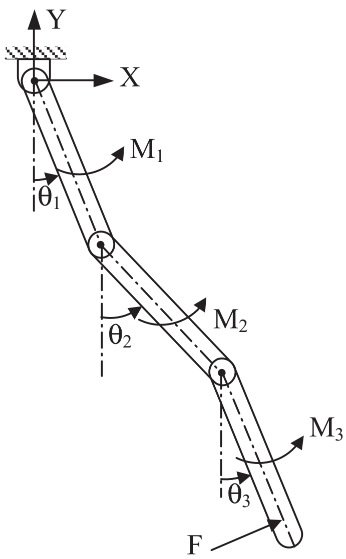

Figure 1.27 Three-degrees-of-freedom system.

Solution to Problem 1.27

The position vector of the point of application of the force F is

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 + \ell_3 \sin\theta_3 \\ -\left(\ell_1 \cos\theta_1 + \ell_2 \cos\theta_2 + \ell_3 \cos\theta_3\right) \end{bmatrix} \tag{1.27.1}$$

where ℓ_1, ℓ_2, and ℓ_3 are the lengths of the rods. The virtual change in the position vector $\mathbf{r}$ is given by

$$\delta\mathbf{r} = \begin{bmatrix} \ell_1 \cos\theta_1\delta\theta_1 + \ell_2 \cos\theta_2\delta\theta_2 + \ell_3 \cos\theta_3\delta\theta_3 \\ \ell_1 \sin\theta_1\delta\theta_1 + \ell_2 \sin\theta_2\delta\theta_2 + \ell_3 \sin\theta_3\delta\theta_3 \end{bmatrix} \tag{1.27.2}$$

or

$$\delta\mathbf{r} = \begin{bmatrix} \delta r_1 \\ \delta r_2 \end{bmatrix} \begin{bmatrix} \ell_1 \cos\theta_1 & \ell_2 \cos\theta_2 & \ell_3 \cos\theta_3 \\ \ell_1 \sin\theta_1 & \ell_2 \sin\theta_2 & \ell_3 \sin\theta_3 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} \tag{1.27.3}$$

The virtual work of the component M_i of the moment is $M_i\delta\theta_i$. Hence, the virtual work δW of the force vector $\mathbf{F}$ and the moment $\mathbf{M}$ are

$$\delta W = \mathbf{F}^T\delta\mathbf{r} + \mathbf{M}^T\delta\theta \tag{1.27.4}$$

where

$$\delta\theta = \begin{bmatrix} \delta\theta_1 & \delta\theta_2 & \delta\theta_3 \end{bmatrix}^T$$

The virtual work δW is then given by

$$\delta W = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \ell_1 \cos\theta_1 & \ell_2 \cos\theta_2 & \ell_3 \cos\theta_3 \\ \ell_1 \sin\theta_1 & \ell_2 \sin\theta_2 & \ell_3 \sin\theta_3 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} + \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix}$$
$$= \begin{bmatrix} M_1 + \ell_1(F_1\cos\theta_1 + F_2\sin\theta_1) & M_2 + \ell_2(F_1\cos\theta_2 + F_2\sin\theta_2) & M_3 + \ell_3(F_1\cos\theta_3 + F_2\sin\theta_3) \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} \tag{E.5}$$

or

$$\delta W = \begin{bmatrix} Q_1 & Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} = Q^T \delta\theta \tag{E.6}$$

where the generalized forces Q_1, Q_2, and Q_3 in terms of the generalized coordinates θ_1, θ_2, and θ_3 are given by

$$\begin{aligned} Q_1 &= M_1 + \ell_1(F_1\cos\theta_1 + F_2\sin\theta_1) \\ Q_2 &= M_2 + \ell_2(F_1\cos\theta_2 + F_2\sin\theta_2) \\ Q_3 &= M_3 + \ell_3(F_1\cos\theta_3 + F_2\sin\theta_3) \end{aligned} \tag{E.7}$$

Problem 1.28

Derive the equations of motion for the system shown in Figure 1.28. The coupling springs are unstressed when the pendulums are in the vertical position.

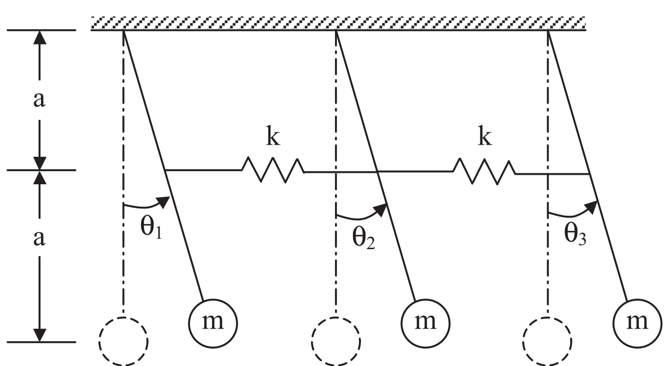

Figure 1.28 Three-degrees-of-freedom system.

Solution to Problem 1.28

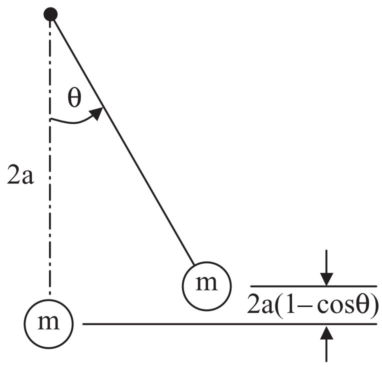

Figure 1.28 (a) Free body diagram.

The kinetic energy of the system is

$$KE = T = \frac{1}{2}m\left(2a\dot{\theta}_1\right)^2 + \frac{1}{2}m\left(2a\dot{\theta}_2\right)^2 + \frac{1}{2}m\left(2a\dot{\theta}_3\right)^2$$

where 2a is the length of all the pendulums, and the θs are the angular displacements.

The potential energy of the system is

$$PE = V = 2mga\left[\left(1 - \cos\theta_1\right) + \left(1 - \cos\theta_2\right) + \left(1 - \cos\theta_3\right)\right]$$
$$+ \frac{1}{2}k\left[\left(a\theta_2 - a\theta_1\right)^2 + \left(a\theta_3 - a\theta_2\right)^2\right]$$

Applying Lagrange's equations gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

where q_i are the generalized coordinates. They are θ_1, θ_2, and θ_3 for this system.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = 4ma^{2\ddot{\theta}_1}$$

$$\frac{\partial T}{\partial \theta_1} = 0$$

$$\frac{\partial V}{\partial \theta_1} = 2mga\sin\theta_1 - ka\left(a\theta_2 - a\theta_1\right)$$

$$4ma^{2\ddot{\theta}_1} + \left(2mga + ka^2\right)\theta_1 - ka^2\theta_2 = 0$$

This is the equation of motion for the first mass, assuming small angles for which $\sin\theta = \theta$.

Similarly,

$$4ma^{2\ddot{\theta}_2} + \left(2mga + ka^2\right)\theta_2 - ka^2\theta_1 - ka^2\theta_3 = 0$$

$$4ma^{2\ddot{\theta}_3} + \left(2mga + ka^2\right)\theta_3 - ka^2\theta_2 = 0$$

PROBLEM 1.29

Derive the differential equations governing the motion of the system shown in Figure 1.29 using Lagrange's equation and the generalized coordinates θ_1, θ_2, θ_3, and θ_4.

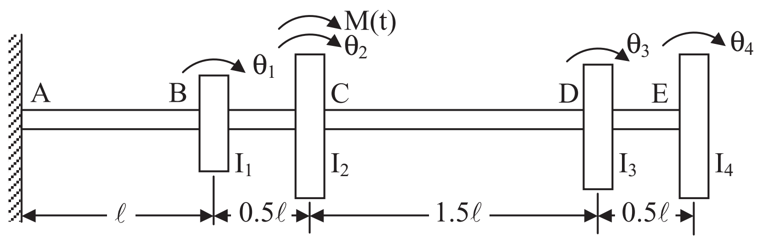

Figure 1.29 Four-degrees-of-freedom system.

Solution to Problem 1.29

Denoting $k = k_{AB} = \dfrac{JG}{\ell}$, we have

$$k_{BC} = \frac{JG}{0.5\ell} = 2k$$

$$k_{CD} = \frac{JG}{0.5\ell} = \frac{2}{3}k$$

$$k_{DE} = \frac{JG}{0.5\ell} = 2k$$

The kinetic energy of the system is

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}I_4\dot{\theta}_4^2$$

The potential energy of the system is

$$V = \frac{1}{2}k\theta_1^2 + \frac{1}{2}(2k)(\theta_2 - \theta_1)^2 + \frac{1}{2}\left(\frac{2}{3}k\right)(\theta_3 - \theta_2)^2 + \frac{1}{2}(2k)\theta_4^2$$

The Lagrangian equation is

$$L = T - V = \frac{1}{2}\left[I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}I_4\dot{\theta}_4^2 - k\theta_1^2 - (2k)(\theta_2 - \theta_1)^2 - \frac{2}{3}k(\theta_3 - \theta_2) - 2k\theta_4^2\right]$$

Applying Lagrange's equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}(I_1\dot{\theta}_1) + k\theta_1 + 2k(\theta_2 - \theta_1)(-1) = 0$$

or

$$I_1\ddot{\theta}_1 + 3k\theta_1 - 2k\theta_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{dt}\left(I_2\dot{\theta}_2\right) + 2k\left(\theta_2 - \theta_1\right) + \frac{2}{3}k\left(\theta_3 - \theta_2\right)\left(-1\right) = 0$$

or

$$I_2\ddot{\theta}_2 - 2k\theta_1 + \frac{8}{3}k\theta_2 - \frac{2}{3}k\theta_3 I_1\ddot{\theta}_1 + 3k\theta_1 - 2k\theta_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} = 0$$

$$\frac{d}{dt}\left(I_3\dot{\theta}_3\right) + \frac{2}{3}k\left(\theta_3 - \theta_2\right) + 2k\left(\theta_4 - \theta_3\right)\left(-1\right) = 0$$

or

$$I_3\ddot{\theta}_3 - \frac{2}{3}k\theta_2 + \frac{8}{3}k\theta_3 - 2k\theta_4 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_4}\right) - \frac{\partial L}{\partial \theta_4} = 0$$

$$\frac{d}{dt}\left(I_4\dot{\theta}_4\right) + 2k\left(\theta_4 - \theta_3\right) = 0$$

or

$$I_4\ddot{\theta}_4 - 2k\theta_3 + 2k\theta_4 = 0$$

The work done by the external moment when virtual rotations are introduced is given by

$$\delta W = M\left(T\right)\delta\theta^2$$

The components of the force vector therefore are given by

$$\begin{aligned} F_1 &= 0 \\ F_2 &= M\left(t\right) \\ F_3 &= 0 \\ F_4 &= 0 \end{aligned}$$

The differential equations of motion can be arranged in matrix form as

$$\begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & I_4 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + \begin{bmatrix} 3k & -2k & 0 & 0 \\ -2k & \frac{8}{3}k & \frac{-2}{3}k & 0 \\ 0 & \frac{-2}{3}k & \frac{8}{3}k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M(t) \\ 0 \\ 0 \end{Bmatrix}$$

We observe that the mass matrix is diagonal and that the system is not dynamically coupled. Similarly, the system is statically coupled because the stiffness matrix is not diagonal.

Chapter 2

Analysis of Dynamic Systems

Problem 2.1

Figure 2.1 shows a simple pendulum attached to a spring of stiffness k.

a. Obtain the differential equation of motion for the system.

b. Determine the natural frequency of oscillation.

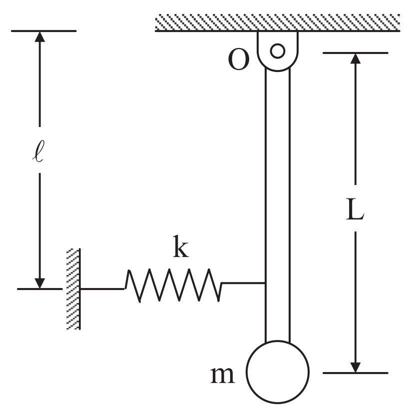

Figure 2.1 Simple pendulum.

Solution to Problem 2.1

a. The differential equation of motion [see Figure 2.1(a)] is given by

$$\sum M_0 = mL^{2\theta} = -k\ell\sin\theta\ \ell\cos\theta - mgL\sin\theta \tag{2.1.1}$$

or

$$mL^{2\theta} + -k\ell^2\sin\theta\ \cos\theta + mgL\sin\theta = 0$$

For small θ, $\sin\theta \approx \theta$ and $\cos\theta = 1$.

Hence,

$$\ddot{\theta} + \left[\frac{k}{m}\frac{\ell^2}{L^2} + \frac{g}{L}\right]\theta = 0 \tag{2.1.2}$$

The natural frequency of oscillation from Eq. 2.2 is

$$\omega_n = \sqrt{\frac{k\ell^2}{mL^2} + \frac{g}{L}} \tag{2.1.3}$$

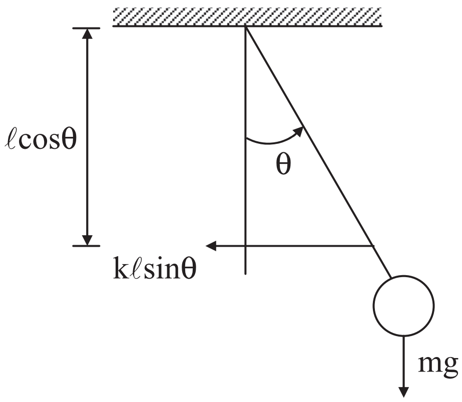

Figure 2.1 (a) Free body diagram.

Problem 2.2

Figure 2.2 shows a uniform bar of mass m and length L that rotates at a constant angular velocity ω about a vertical axis. If θ is the angle between the vertical axis and the bar, as shown in the figure, determine the following:

a. The equilibrium positions expressed in terms of the constant angle θ_0

b. The differential equation of motion for small motions about θ_0

c. The stability criterion for each equilibrium position based on the requirement that the motion θ_1 must be harmonic

d. The natural frequency of the oscillation θ_1 for the stable cases

e. The natural frequency for very large ω

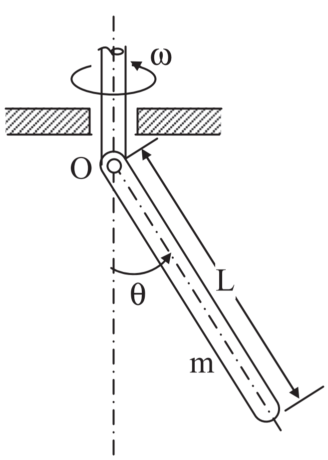

Figure 2.2 Dynamic system.

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Solution to Problem 2.2

a. Taking moments about O,

$$\sum M_0 = mg\frac{L}{2}\sin\theta + \frac{1}{2}m\omega^2 L\sin\theta\frac{2}{3}L\cos\theta = I_{0\ddot{\theta}} \tag{2.2.1}$$

where $I_0 = mL^2/3$.

Hence,

$$\frac{mL^2\ddot{\theta}}{3} + \frac{mgL}{2}\sin\theta - \frac{1}{3}m\omega^2L^2\sin\theta\cos\theta = 0 \tag{2.2.2}$$

In equilibrium position, $\theta = \theta_0$ and $\dot{\theta} = \ddot{\theta} = 0$.

Therefore,

$$mL\left(\frac{g}{2} - \frac{1}{3}\omega^2 L\cos\theta_0\right)\sin\theta_0 = 0 \tag{2.2.3}$$

Equilibrium positions are

$$\theta_0 = 0,\ \ \pi,\ \cos^{-1}\left(\frac{3g}{2\omega^2 L}\right) \tag{2.2.4}$$

b. Let $\theta(t) = \theta_0 + \theta_1(t)$ and $\theta_1(t)$ is small, such that

$$\begin{aligned}\sin\theta &\approx \sin\theta_0 + \theta_1\cos\theta_0\\ \cos\theta &\approx \cos\theta_0 - \theta_1\sin\theta_0\end{aligned} \tag{2.2.5}$$

Substituting Eq. 2.2.5 in the equation of motion, Eq. 2.2.2, and neglecting the second-order terms in θ_1, we obtain

$$\ddot{\theta}_1 + \left(\frac{3g}{2L}\cos\theta_0 - \omega^2\cos 2\theta_0\right)\theta_1 = 0 \tag{2.2.6}$$

c. For stability,

$$\frac{3g}{2L}\cos\theta_0 - \omega^2\cos 2\theta_0 > 0$$

Equilibrium position $\theta_0 = 0$ is stable if $\omega^2 < 3g/2L$.

Equilibrium position $\theta_0 = \cos^{-1}\left(3g/2\omega^2 L\right)$ is stable if $\omega^2 > 3g/2L$.

Equilibrium position $\theta_0 = \pi$ is unstable.

d. To determine the natural frequency of the oscillation θ_1 for the stable cases,

$$\theta_0 = 0 \quad \text{and} \quad \omega_n = \sqrt{\frac{3g}{2L} - \omega^2} \tag{2.2.7}$$

$$\theta_0 = \cos^{-1}\left(\frac{3g}{2\omega^2 L}\right) \quad \text{and} \quad \omega_n = \sqrt{\omega^2 - \frac{9g^2}{4\omega^2 L^2}} \tag{2.2.8}$$

e. For very large ω, $\omega_n = \omega$ and $\theta_0 = \pi/2$, which indicates that the bar oscillates about the horizontal position.

Problem 2.3

Figure 2.3 shows a torsional vibration system in which the disk, which has a mass moment of inertia about its axis I_G, is fastened to shafts of different d_1 and d_2, and the shafts are fixed at both ends. Obtain the following:

a. The differential equation of angular motion

b. The natural frequency of angular vibration

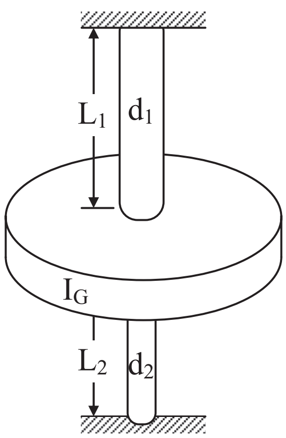

Figure 2.3 Torsional vibrating system.

Solution to Problem 2.3

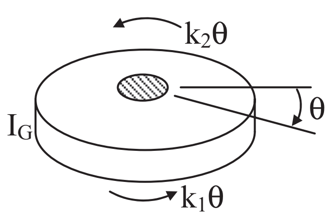

Figure 2.3 (a) Free body diagram.

a. For the differential equation of angular motion,

$$K_{t_1} = \frac{T_1}{\theta} = \frac{GJ_1}{L_1} = \frac{\pi}{32}\frac{G}{L_1}d_1^4 \tag{2.3.1}$$

$$K_{t_2} = \frac{T_2}{\theta} = \frac{GJ_2}{L_2} = \frac{\pi}{32}\frac{G}{L_1}d_1^4 \tag{2.3.2}$$

$$I_G\ddot{\theta} = -k_{t_1}\theta - k_{t_2}\theta \tag{2.3.3}$$

The differential equation of angular motion is

$$\ddot{\theta} + \left(\frac{k_{t_1} + k_{t_2}}{I_G} \right) \theta = 0 \tag{2.3.4}$$

b. The natural frequency of angular vibration is

$$\omega_n = \sqrt{\frac{k_{t_1} + k_{t_2}}{I_G}} \text{ rad/s} \tag{2.3.5}$$

or

$$\begin{aligned} f_n &= \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k_{t_1} + k_{t_2}}{I_G}} \\ &= \frac{1}{2\pi} \sqrt{\frac{\pi G}{32 I_G} \left(\frac{d_1^4}{L_1} + \frac{d_2^4}{L_2} \right)} \\ &= \frac{1}{8} \sqrt{\frac{G}{2\pi I_G} \left(\frac{d_1^4}{L_1} + \frac{d_2^4}{L_2} \right)} \end{aligned} \tag{2.3.6}$$

Problem 2.4

Determine the natural frequency of angular oscillations of the simple pendulum shown in Figure 2.4 if the following are true:

a. The mass of the rod m_r is negligible.

b. The mass of the rod m_r is not negligible.

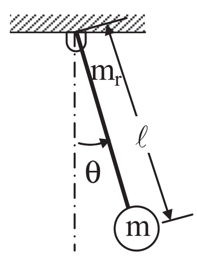

Figure 2.4 Simple pendulum.

Solution to Problem 2.4

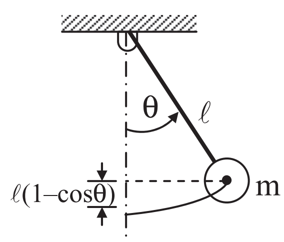

Figure 2.4 (a) Free body diagram.

a. We use the energy method here as

$$d/dt\left(T + V\right) = 0$$

For small rotation of the mass about the pivot,

Kinetic energy: $T = \frac{1}{2} m\, \dot{x}^2 = \frac{1}{2} m\left(\ell\dot{\theta}\right)^2$

Potential energy: $V = mg\ell\left(1 - \cos\theta\right)$ (2.4.1)

$$\frac{d}{dt}\left(T + V\right) = m\ell^{2\dot{\theta}\,\ddot{\theta}} + mg\ell \sin\theta\,\dot{\theta} = 0 \qquad (2.4.2)$$

or

$$\ddot{\theta} + \left(\frac{g}{\ell}\right)\sin\theta = \theta \qquad (2.4.3)$$

For small angles of oscillation, $\sin\theta \approx \theta$, and the equation of motion when the mass of the rod is neglected is

$$\ddot{\theta} + \left(\frac{g}{\ell}\right)\sin\theta = \theta \qquad (2.4.4)$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{g}{\ell}}\ \text{rad/s}$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}\ \text{Hz} \qquad (2.4.5)$$

b. The mass of the rod m_r acts through the center of the rod.

Kinetic energy: T = kinetic energy of the mass + kinetic energy of the rod

$$\frac{1}{2} m\left(\ell\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right) m_r\left(\ell\dot{\theta}\right)^2 \qquad (2.4.6)$$

Potential energy: V = potential energy of the mass + potential energy of the rod

$$mg\ell\left(1 - \cos\theta\right) + m_r g\left(\frac{\ell}{2}\right)\left(1 - \cos\theta\right) \qquad (2.4.7)$$

$$\frac{d}{dt}\left(T + V\right) = \left(m + \frac{m_r}{3}\right)\ell^{2\dot{\theta}\,\ddot{\theta}} + g\ell\left(m + \frac{m_r}{2}\right)\sin\theta\,\dot{\theta} \qquad (2.4.8)$$

or

$$\ddot{\theta} + \left[\frac{m + \frac{m_r}{2}}{m + \frac{m_r}{3}} \sin\theta \right] \left(\frac{g}{\ell} \right) \theta = 0 \qquad (2.4.9)$$

For small angles of oscillation, $\sin\theta \approx \theta$, and Eq. 2.4.4 becomes

$$\ddot{\theta} + \left[\frac{m + \frac{m_r}{2}}{m + \frac{m_r}{3}} \right] \left(\frac{g}{\ell} \right) \theta = 0 \qquad (2.4.10)$$

$$\omega_n = \sqrt{\frac{m + \frac{m_r}{2}}{m + \frac{m_r}{3}} \left(\frac{g}{\ell} \right)} \text{ rad/s} \qquad (2.4.11)$$

If $m_r << m$, then,

$$\sqrt{\frac{m + \frac{m_r}{2}}{m + \frac{m_r}{3}}} \approx 1$$

and

$$\omega_n = \sqrt{\left(\frac{g}{\ell} \right)} \qquad (2.4.12)$$

which is the same as Eq. 2.4.5 of Part a of this problem.

Problem 2.5

A single-degree-of-freedom vibrating system has the values m = 4 lb^2/in. (73 kg), k = 1600 lb/in. (28.02×10^4 N/m), and $\mu_k = 0.1$. Calculate the decay per cycle and the number of half-cycles until oscillation stops if the initial conditions are $x(0) = x_0 = 1.1$ in. (279.4 mm) and $\dot{x}(0) = 0$.

Solution to Problem 2.5

The decay per cycle is

$$4f_d = 4\frac{F_d}{k} = 4\frac{\mu_k mg}{k} = 4\frac{(0.1)(4)(32.2)(12)}{1600} = 4(0.0966) = 0.3864 \text{ in.} \qquad (2.5.1)$$

Moreover, n must be the smallest integer satisfying the inequality

$$1.1 - (2n - 1)(0.0966) < 2(0.0966) \qquad (2.5.2)$$

from which we conclude that the oscillation stops after the half-cycle n = 6 with m in the position $x(t_6) = x_0 - 12f_d = 1.2 - 12(0.0966) = 0.0408$ in.

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Problem 2.6

A spring-mounted body of mass m is connected to the wheel by a linear spring of stiffness k, and a viscous damper of damping coefficient c moves with a velocity on a road surface, as shown in Figure 2.6. The road surface has a wavelength of L and an amplitude of h. Obtain a relationship for the ratio of amplitudes of the absolute vertical displacement of the body to the road surface undulations.

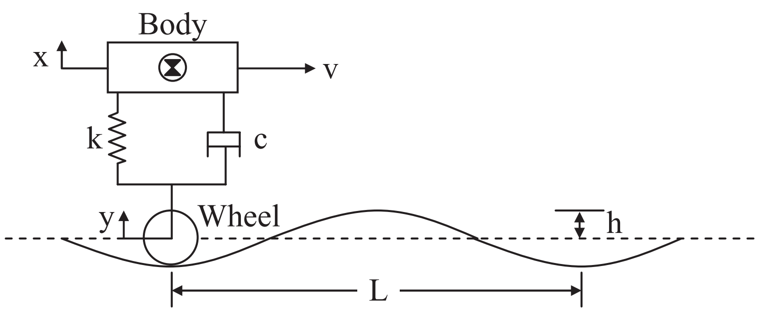


Figure 2.6 Road vehicle system.

Solution to Problem 2.6

The system can be modeled as shown in Figure 2.6(a).

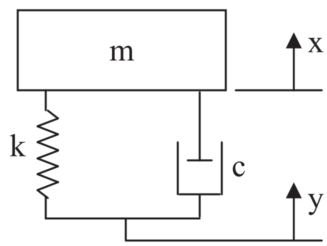


Figure 2.6 (a) Free body diagram.

where

$$y = h\cos\frac{2\pi z}{L}$$

and

$$z = vt$$

Therefore,

$$y = h\cos\left(\frac{2\pi v}{L}\right)t = h\cos Vt$$

where

$$V = \frac{2\pi v}{L} \tag{2.6.1}$$

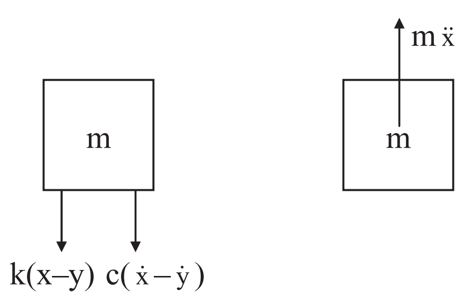


Figure 2.6 (b) Free body diagram.

Hence, the equation of motion can be written as

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}) \tag{2.6.2}$$

or

$$m\ddot{x} + c\dot{x} + kx = cy + ky \tag{2.6.3}$$

Because y = h cos V, Eq. 2.6.3 becomes

$$m\ddot{x} + c\dot{x} + kx = \sqrt{\left[k^2 + (cV)^2\right]}h\sin(Vt + \phi) \tag{2.6.4}$$

Therefore, if

$$x = X_0 \sin(Vt + \alpha) \tag{2.6.5}$$

then,

$$X_0 = \frac{h\sqrt{\left[k^2 + (cV)^2\right]}}{\sqrt{\left[\left(k - mV^2\right)^2 + (cV)^2\right]}} \tag{2.6.6}$$

Hence, the ratio of amplitude of the absolute vertical displacement of the body to the road surface undulations is given by

$$\frac{X_0}{h} = \frac{X_0}{h} = \frac{\sqrt{\left[k^2 + \left(\frac{2\pi v}{L}c\right)^2\right]}}{\sqrt{\left(k - \left(\frac{2\pi v}{L}\right)^2 m\right)^2 + \left(\frac{2\pi v}{L}c\right)^2}} \tag{2.4.7}$$

Problem 2.7

A machine of 13.7 slug (200 kg) mass is supported on four parallel springs of total stiffness 4283 lb/in. (750 kN/m) and has an unbalanced rotating component that results in a disturbing force of 78.68 lb_f (350 N) at an operating speed of 2121 rpm. If the damping factor is 0.20, determine the following:

a. The amplitude of motion due to the unbalance

b. The transmissibility

c. The transmitted force

Solution to Problem 2.7

The statistical deflection of the system is

$$\delta_{st} = \frac{200 \times 9.81}{700 \times 10^3} = 2802 \times 10^{-3}\ \text{m} = 2.802\ \text{mm}$$

The natural frequency of the system is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{2.802 \times 10^{-3}}} = 9.417 \text{ Hz}$$

$$r = \frac{f}{f_n} = \frac{\left(\frac{2121}{60}\right)}{9.417} = 3.754$$

a. The amplitude of motion due to the unbalance is obtained from

$$X = \frac{\left(\frac{F_0}{k}\right)}{\sqrt{\left(1-r^2\right)^2 + \left[2\zeta r\right]^2}} = \frac{\frac{350}{750 \times 10^3}}{\sqrt{\left(1-3.754^2\right)^2 + \left[2(0.2)(3.754)\right]^2}}$$

$$= 3.791 \times 10^{-5} \text{ m} = 0.03791 \text{ mm}$$

b. The transmissibility is

$$\beta_r = \frac{\sqrt{1+\left(2\zeta r\right)^2}}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} = \frac{\sqrt{1+\left[2(0.2)(3.754)\right]^2}}{\sqrt{\left(1-3.754^2\right)^2 + \left[2(0.2)(3.754)\right]^2}} = 0.137$$

c. The transmitted force is the disturbing force times the transmissibility.

$$F_t = F_0\beta_r = 350(0.137) = 47.95 \text{ N}$$

Problem 2.8

A simplified model of a passenger car traveling over a road and vibrating in the vertical direction is shown in Figure 2.8. The mass of the vehicle is 68.52 slugs (1000 kg), the spring stiffness of the suspension system is 2855.08 lb/in. (500 kN/m), and there is a damping factor of $\xi = 0.5$. The road surface is assumed to vary sinusoidally with an amplitude of 0.984 in. (0.025 m). Determine the displacement amplitude of the vehicle.

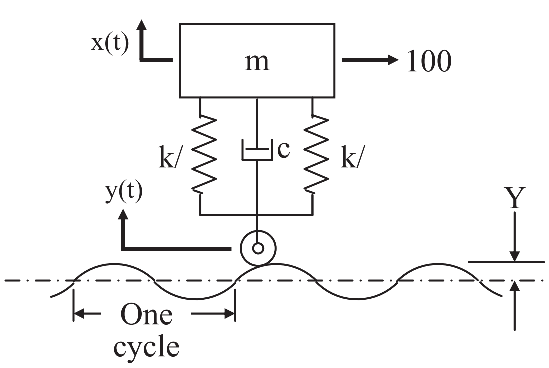


Figure 2.8 Passenger car traveling on a road.

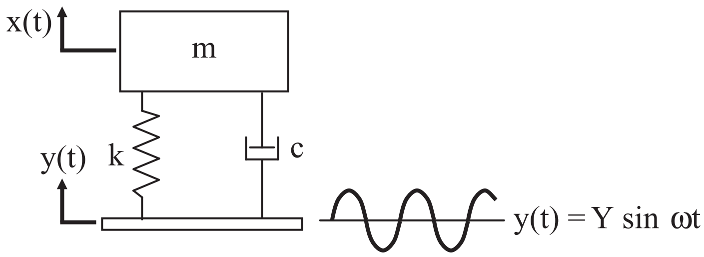


Figure 2.8 (a) Vehicle moving over a rough road.

Solution to Problem 2.8

The frequency ω of the base excitation can be found by dividing the vehicle speed by the length of one cycle of road roughness.

$$\omega = 2\pi f = 2\pi\left(\frac{122.50 \times 1000}{3600}\right)\frac{1}{6} = 35.62 \text{ rad/s}$$

The natural frequency of the vehicle is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{500 \times 10^3}{1000}\right)^{1/2} = 22.36 \text{ rad/s}$$

Hence, the frequency ratio r is

$$r = \frac{\omega}{\omega_n} = \frac{29.0887}{22.36} = 1.593$$

The amplitude ratio can be found from Eq. 3.63 as

$$\begin{aligned}\frac{X}{Y} &= \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right\}^{1/2} \\ &= \left\{\frac{1+(2\times 0.5\times 1.539)^2}{(1-1.593^2)^2+(2\times 0.5\times 1.593)^2}\right\}^{1/2} \\ &= 0.8493\end{aligned}$$

Thus, the displacement amplitude of the vehicle is given by

$$X = 0.8393, \quad Y = 0.8493(0.025) = 0.02123 \text{ m} \quad \text{or} \;\; 21.23 \text{ m}$$

Problem 2.9

Figure 2.9(a) shows the simplified model of a road vehicle suspension system. The body of a 68.52-slug (1000-kg) vehicle is connected to the wheels through a suspension system with a spring stiffness of 34260 lb/ft (5×10^5 N/m) in parallel with a viscous damper of damping coefficient 346.2 lb.s/ft (5000 Ns/m). The wheels are assumed to follow the road contour given in Figure 2.9(b). If the vehicle travels at a uniform speed of 164.04 ft/s (50 m/s), find the acceleration amplitude of the vehicle. Given are L = 16.4 ft (5 m) and c = 0.066 ft (0.02 m).

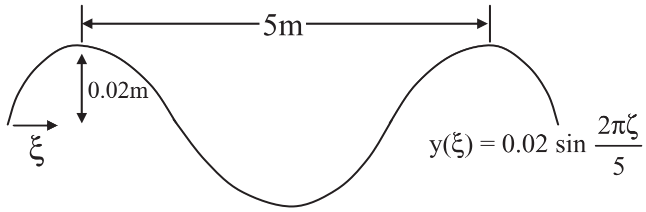


Figure 2.9 (a) Simplified model of a road vehicle suspension system.

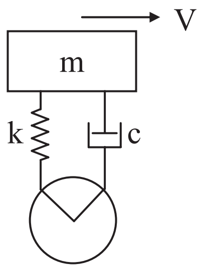


Figure 2.9 (b) Road contour.

Solution to Problem 2.9

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5 \times 10^5}{1000}} = 22.36 \text{ rad/s}$$

The damping ratio $\zeta =$ is

$$\zeta = \frac{c}{2m\omega_n} = \frac{5000}{2(1000)(22.36)} = 0.1118$$

The road contour is

$$y(\zeta) = 0.02\sin(0.4\pi\zeta) \tag{2.9.1}$$

The vehicle travels at a uniform velocity, $\xi = Vt$. Hence, the time-dependent vertical displacement of the wheel is

$$y(t) = 0.02\sin[0.4\pi Vt] \tag{2.9.2}$$

If it is assumed that the wheel follows the road contour, it acts as a harmonic base displacement for the body of the vehicle. The frequency of the displacement is

$$\omega = 0.4\pi V = 0.4\pi(50) = 62.83 \text{ rad/s}$$

The frequency ratio is

$$R = \frac{\omega}{\omega_n} = \frac{62.83}{22.36} = 2.81$$

The amplitude of absolute displacement of the vehicle is

$$x = 0.02\sqrt{\frac{1 + [2(0.1118)(2.81)]^2}{(1 - 2.81^2)^2 + [2(0.1118)(2.81)]^2}} = 3.40939 \times 10^{-3} \text{ m}$$

The acceleration amplitude of the vehicle is

$$A = \omega^2 X = (62.83)^2 (3.40939 \times 10^{-3}) = 13.4589 \text{ m/s}^2$$

Problem 2.10

The support of a simple pendulum shown in Figure 2.10 has a specified motion $y = Y_0 \sin \omega t$. If small oscillations of the system are assumed, determine the following:

a. The differential equation of motion

b. The steady-state solution

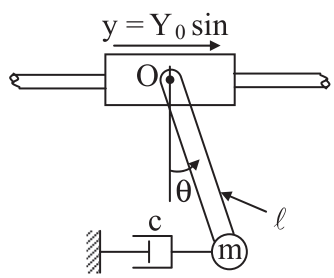


Figure 2.10 Simple pendulum.

Solution to Problem 2.10

For small angular oscillations, the displacement of the mass m in the horizontal direction is given by

$$x = y + \ell\theta = Y_0 \sin \omega t + \ell\theta \tag{2.10.1}$$

The velocity and acceleration of the mass are

$$\dot{x} = \dot{y} + \ell\dot{\theta} = \omega Y_0 \sin \omega t + \ell\dot{\theta} \tag{2.10.2}$$

$$\ddot{x} = \ddot{y} + \ell\ddot{\theta} = -\omega Y_0 \sin \omega t + \ell\ddot{\theta} \tag{2.10.3}$$

Let R_x and R_y denote the reaction forces, as shown in Figure 2.10(a).

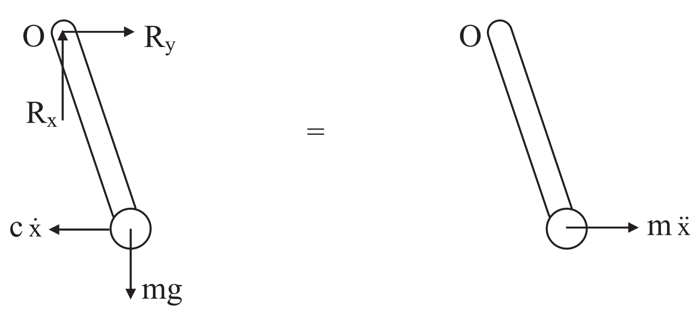


Figure 2.10 (a) Free body diagrams.

Taking the moments of the applied and inertia forces about O, we obtain the dynamic equation

$$-mg\ell \sin\theta - c\dot{x}\ell \cos\theta = m\ddot{x}\ell \cos\theta \tag{2.10.4}$$

For small angular oscillation, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.

Hence,

$$m\ddot{x}\ell + c\ell + mg\ell\theta = 0 \tag{2.10.5}$$

Using Eqs. 2.10.1 through 2.10.3 for x, $\dot{x}$, and $\ddot{x}$, we obtain

$$m\left(-\omega^2 Y_0 \sin\omega t + \ell\ddot{\theta}\right)1 + c\left(\omega Y_0 \cos\omega t + \ell\dot{\theta}\right)\ell + mg\ell\theta = 0 \tag{2.10.6}$$

Equation 2.10.6 can be written as

$$\begin{aligned} m\ell^{2\ddot{\theta}} + c\ell^{2\dot{\theta}} + mg\ell\theta &= m\omega^2 Y_0 \ell \sin\omega t - c\omega Y_0 \ell \cos\omega t \lim_{x\to\infty} \\ &= \omega Y_0 \ell \left[m\omega \sin\omega t - c\cos\omega t\right] \\ &= \omega Y_0 \ell \sqrt{\left(m\omega\right)^2 + c^2} \sin\omega t\left(\omega t - \psi_b\right) \end{aligned} \tag{2.10.7}$$

which can be written as

$$m_e\ddot{\theta} + c_e\dot{\theta} + k_e\theta = F_e \sin\left(\omega t - \psi_b\right) \tag{2.10.8}$$

where

$$\begin{aligned} m_e &= m\ell^2 \\ k_e &= mg\ell \\ F_e &= \omega Y_0 \ell \sqrt{\left(m\omega\right)^2 + c^2} \\ \psi_b &= \tan^{-1}\left(\frac{c}{m\omega}\right) \end{aligned} \tag{2.10.9}$$

Hence, the steady-state solution can be expressed as

$$\theta_p = \frac{F_e/k_e}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}} \sin\left(\omega t - \psi - \psi_b\right) \tag{2.10.10}$$

$$\begin{aligned} r &= \frac{\omega}{\omega_n} \\ \omega_n &= \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{mg\ell}{m\ell^2}} = \sqrt{\frac{g}{\ell}} \\ \xi &= \frac{c_e}{C_c} - \frac{c\ell^2}{C_c} \\ C_c &= 2m_e\omega_n = 2m\ell^2\sqrt{\frac{g}{\ell}} \\ \psi &= \tan^{-1}\left(\frac{2r\xi}{1 - r^2}\right) \end{aligned} \tag{2.10.11}$$

Problem 2.11

Figure 2.11 shows the simplified two-degrees-of-freedom model of a passenger car. Determine the normal modes of vibration with the following data:

$W = 3000$ lb (13.375 kN)

$\ell_1 = 4$ ft (1.22 m)

$k_1 = 2000$ lb/ft (29188 N/m)

$J_c = \dfrac{W}{g} r^2$

$\ell_2 = 6$ ft

$k_2 = 2500$ lb/ft

$r = 3.5$ ft (1.067 m)

$\ell = 10$ ft (3.05 m)

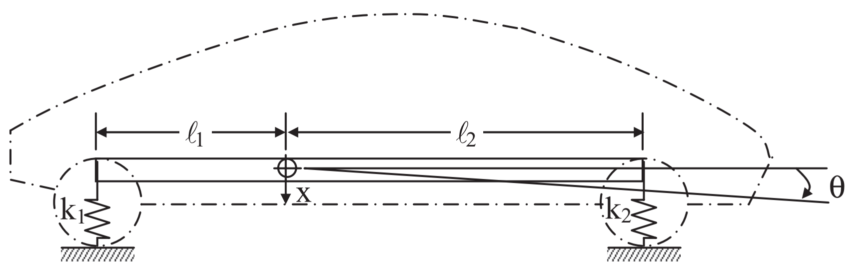


Figure 2.11 Simplified two-degrees-of-freedom model of a passenger car.

Solution to Problem 2.11

The equations of motion are

$$m\ddot{x} + k_1\left(x - \ell_1\theta\right) + k_2\left(x + \ell_2\theta\right) = 0 \tag{2.11.1}$$

$$J\ddot{\theta} - k_1\left(x - \ell_1\theta\right)\ell_1 + k_2\left(x + \ell_2\theta\right)\ell_2 = 0 \tag{2.11.2}$$

Equations 2.11.1 and 2.11.2 indicate static coupling.

If harmonic motion is assumed, we can write

$$\begin{bmatrix} \left(k_1 + k_2 - \omega^2 m\right) & -\left(k_1\ell_1 - k_2\ell_2\right) \\ -\left(k_1\ell_1 - k_2\ell_2\right) & \left(k_1\ell_1^2 + k_2\ell_2^2 - \omega^2 J_c\right) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.11.3}$$

From the determinant of the matrix in Equation 2.11.3, the two natural frequencies computed are

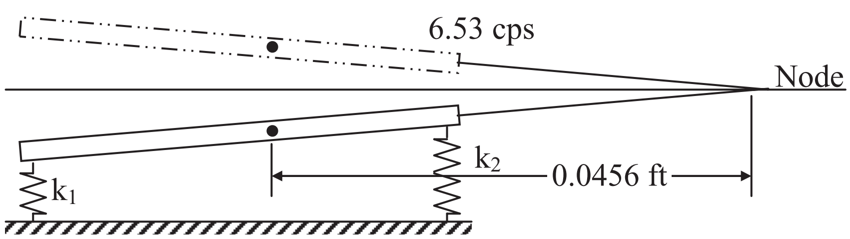
$$\begin{aligned} \omega_1 &= 41.02 \text{ rad/s} = 6.53 \text{ cps} \\ \omega_2 &= 113.21 \text{ rad/s} = 18.01 \text{ cps} \end{aligned} \tag{2.11.4}$$

The amplitude ratios for the two frequencies are

$$\left(\frac{x}{\theta}\right)_{\omega_1} = 0.0456 \text{ ft/rad} = 0.00955 \text{ in./deg}$$
$$\left(\frac{x}{\theta}\right)_{\omega_2} = 0.00584 \text{ ft/rad} = 0.0431 \text{ in./deg} \quad (2.11.5)$$

The mode shapes are illustrated in Figure 2.11(a).

The first mode, $\omega_1 = 41.02$ rad/s, is largely vertical translation with very small rotation, whereas the second mode, $\omega_2 = 113.21$ rad/s, is mostly rotation, suggesting that we could have made a rough approximation for these modes as two one-degree-of-freedom systems.



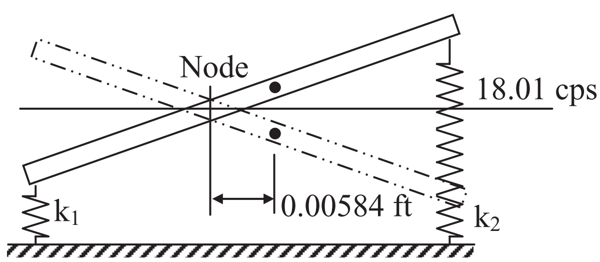


Figure 2.11 (a) First mode shape, and (b) second mode shape.

Problem 2.12

While referring to Figure 2.12, determine the principal modes, including the locations of the modes, by using the following data:

$\ell_1 = 60$ in. (1.53 m)

W = 3000 lb (13.375 kN)

$\ell_2 = 40$ in. (1.02 m)

$k_1 = 300$ lb/in. (52.538 N/m)

r = 32 in. (0.82 m)

$k_2 = 200$ lb/in. (35025 N/m)

$\ell = 100$ in. (2.54 m)

Solution to Problem 2.12

$$\beta = \frac{k_1 + k_2}{m} = \frac{300 + 200}{3000/386} = 64.35$$

$$\gamma = \frac{k_2\ell_2 - k_1\ell_1}{m} = \frac{200 \times 40 - 300 \times 60}{3000/386} = -1287$$

$$\eta = \frac{k_1\ell_1^2 + k_2\ell_2^2}{m\,r^2} = \frac{300 \times (60)^2 + 200 \times (40)^2}{(32)^2 \times 3000/386} = 175.95$$

Then,

$$\omega^2 = \frac{\eta + \beta}{2} \mp \sqrt{\left(\frac{\eta - \beta}{2}\right)^2 + \left(\frac{\gamma}{r}\right)^2} = \frac{175.95 + 64.35}{2}$$

$$\sqrt{\left(\frac{175.95 - 64.35}{2}\right)^2 + \left(\frac{-1287}{32}\right)^2} = 120.15 \mp 68.78$$

where

$$\omega_1 = 7.167 \text{ rad/s}$$

and

$$\omega_2 = 13.74 \text{ rad/s}$$

and the amplitude ratios are

$$\frac{A}{B} = \frac{\gamma}{(\eta - \beta)/2 \mp \sqrt{\left[(\eta - \beta)/2\right]^2 + (\gamma/r)^2}} = \frac{-1287}{55.8 \mp 68.78}$$

By taking the −ve sign, the first modal amplitude ratio is obtained.

$$\frac{A_1}{B_1} = 99.15 \text{ in./rad}$$

For the second mode, the +ve sign is taken in the expression for $\frac{A}{B}$.

$$\frac{A_2}{B_2} = -100.33 \text{ in./rad}$$

The first and second mode shapes are shown in Figure 2.12.

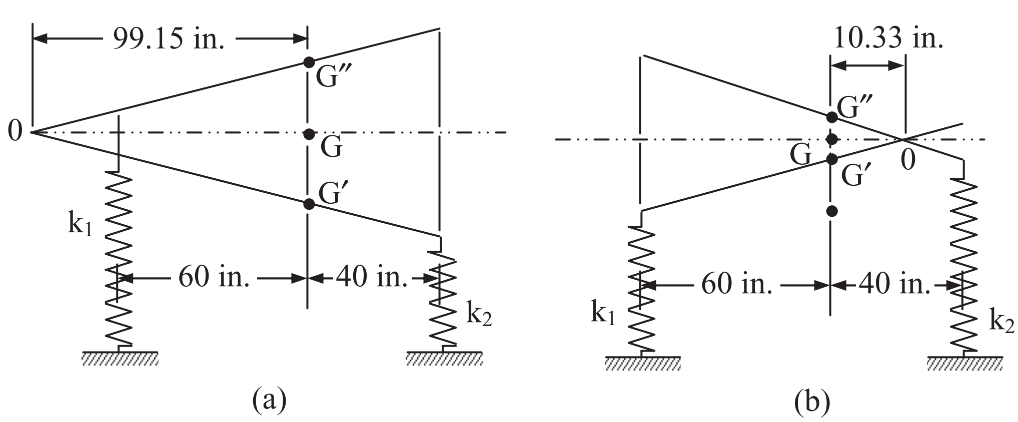


Figure 2.12 (a) First mode shape, and (b) second mode shape.

The node locations may be obtained by considering the tangent defined for a small angular amplitude such as 0.01 rad. The tangent of the angle then also will be 0.01. Thus, for the first mode, if B_1 = 0.01 rad, then A_1 = 0.9915 in., and the distance OG = 0.9915 in./0.01 = 99.15 in., which is the same as the amplitude ratio $\frac{A_1}{B_1}$. Similarly, for the second mode, the distance OG = −10.33 in., which is the same as the amplitude ratio.

Problem 2.13

Determine the pitch (angular motion) and bounce (up and down linear motion) frequency and the location of oscillation centers (nodes) of an automobile with the following data, as shown in Figure 2.13.

Mass = m = 75.37 slugs (1100 kg)

Radius of gyration = r = 3.28 ft (1.0 m)

Distance between the front axle and the center of CG = ℓ_1, = 3.28 ft (1.0 m)

Distance between the rear axle and CG = ℓ_2 = 4.92 ft (1.5 m)

Front spring stiffness = k_f = 91.36 lb/in. (16 kN/m)

Rear spring stiffness = k_r = 119.95 lb/in. (21 kN/m)

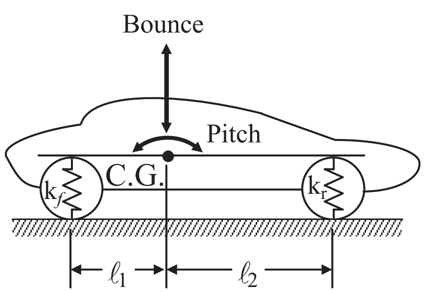


Figure 2.13 Automobile.

Solution to Problem 2.13

If x and θ are used in independent coordinates, the equations of motion are given by Eq. 5.23, with $k_1 = k_f$, $k_2 = k_r$, and $J_0 = mr^2$. For free vibration, we assume a harmonic solution.

$$\begin{aligned} x(t) &= X\cos(\omega t + \phi) \\ \theta(t) &= \Theta\cos(\omega t + \phi) \end{aligned} \tag{2.13.1}$$

By using Eqs. 2.13.1 and 5.23, we obtain

$$\begin{bmatrix} \left(-m\omega^2 + k_1 + k_2\right) & \left(-k_1 l_1 + k_2 l_2\right) \\ \left(-k_1 l_1 + k_2 l_2\right) & \left(-J_0\omega^2 + k_2 l_2^2 + k_1 l_1^2\right) \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.13.2}$$

For the known data, Eq. 2.13.22 becomes

$$\begin{bmatrix} \left(-1100\omega^2 + 33{,}000\right) & 15{,}500 \\ 15{,}500 & \left(-1100\omega^2 + 63{,}250\right) \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.13.3}$$

from which the frequency equation can be derived as

$$\omega^4 - 87.5\omega^2 + 1526.47 = 0 \tag{2.13.4}$$

The natural frequencies can be found from Eq. 2.13.4 as

$$\omega_1 = 4.905 \text{ rad/s} \quad \text{and} \quad \omega_1 = 7.964 \text{ rad/s} \tag{2.13.5}$$

With these values, the ratio of amplitudes can be found for Eq. 2.13.3 as

$$\frac{X^{(1)}}{\Theta^{(1)}} = -2.371 \quad \text{and} \quad \frac{X^{(2)}}{\Theta^{(2)}} = 0.4223 \tag{2.13.6}$$

The node locations can be obtained by noting that the tangent of a small angle is approximately equal to the angle itself. We can find that the distance between the center of gravity (CG) and the node is –2.371 m for ω_1 and 0.4223 m for ω_2.

Problem 2.14

For the three-degrees-of-freedom system shown in Figure 2.14, derive the differential equations of motion using Newton's second law of motion.

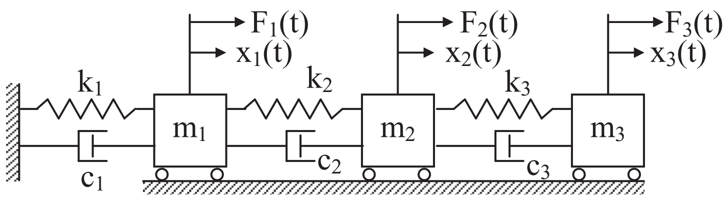


Figure 2.14 Three-degrees-of-freedom vibrating system.

Solution to Problem 2.14

The free body diagram is shown in Figure 2.14(a).

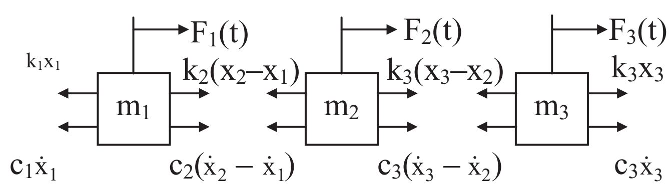


Figure 2.14 (a) Free body diagram.

Application of Newton's second law of motion to masses m_1, m_2, and m_3 gives the following equations of motion:

$$\begin{aligned} F_1 + c_2\left(\dot{x}_2 + \dot{x}_1\right) - k_2\left(\dot{x}_2 - \dot{x}_1\right) - c_1\dot{x}_1 - k_1\dot{x}_1 &= c_1\ddot{q}_1 \\ F_2 + c_3\left(\dot{x}_3 - \dot{x}_2\right) + k_3\left(x_3 - x_2\right) - c_2\left(\dot{x}_2 - \dot{x}_1\right) - k_2\left(x_2 - x_1\right) &= m_2\ddot{q}_2 \\ F_3 - c_3\left(\dot{x}_3 - \dot{x}_2\right) - k_3\left(x_3 - x_2\right) &= m_3\ddot{q}_3 \end{aligned} \tag{2.14.1}$$

These can be rearranged in the form

$$\begin{aligned} m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 &= F_1 \\ m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - c_3\dot{x}_3 - k_2x_1 + (k_3 + k_3)x_2 - k_3x_3 &= F_2 \\ m_3\ddot{x}_3 - c_3\dot{x}_2 + c_3\dot{x}_3 - k_3x_2 + k_3x_3 &= F_3 \end{aligned} \tag{2.14.2}$$

Equation 2.14.2 can be expressed in matrix form as

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t) \tag{2.14.3}$$

where the coefficient matrices are given by

$$m = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{2.14.4}$$

$$c = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{2.14.5}$$

$$k = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{2.14.6}$$

which are clearly symmetric, and M is diagonal.

Problem 2.15

Consider an undamped three-degrees-of-freedom system given by

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix}$$

where $\{F(t)\}$ is a vector of transient excitations.

a. Find the frequency equation and the natural frequencies.

b. Determine the modal vectors and the modal matrix.

c. Verify that the modal vectors are orthogonal relative to the matrices M and K.

Solution to Problem 2.15

a. Homogeneous equations can be written as

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Upon submitting $x_i = X_i \sin(\omega_t + \phi)$, I = 1, 2, and 3, factoring $\sin(\omega_t + \phi)$, and so forth, we get

$$\left(4 - 2\omega^2\right)X_1 - X_2 + 0 = 0$$

$$-X_1 + \left(2 - \omega^2\right)X_2 - X_3 = 0$$

$$0 - X_2 + \left(4 - 2\omega^2\right)X_3 = 0$$

Therefore, the frequency equation is

$$\begin{vmatrix} \left(4 - 2\omega^2\right) & -1 & 0 \\ -1 & \left(2 - \omega^2\right) & -1 \\ 0 & -1 & \left(4 - 2\omega^2\right) \end{vmatrix} = 0$$

which gives $\omega^2 = 1, 2$, and 3.

b. The amplitude ratios are

$$\frac{X_1}{X_2} = \frac{1}{4 - 2\omega^2}$$

$$\frac{X_2}{X_3} = \frac{4 - 2\omega^2}{1}$$

Thus, modal vectors are given as follows:

ω	**1**	$\sqrt{2}$	$\sqrt{3}$
X_1	1	1	1
X_2	2	0	–2
X_3	1	–1	1

Hence, the modal matrix is

$$[u] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

c. To verify that the modal vectors are orthogonal relative to matrices M and K,

$$[u]^T[M][u] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$[u]^T[K][u] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

which are orthogonal matrices.

Problem 2.16

Obtain the eigenvalues of a damped system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\{\ddot{X}\} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\{\dot{x}_1\} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\{X\} = 0$$

Solution to Problem 2.16

Here,

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\tilde{K} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues of the matrix $\tilde{M}^{-1\tilde{K}}$ are

$$\lambda_{1,2=-1.309\pm0.951\,i}$$
$$\lambda_{3,4=-0.191\pm0.588\,i}$$

Hence,

$$\omega_2 = \sqrt{1.309^2 + 0.951^2} = 1.618 \text{ rad/s}$$

$$\omega_1 = \sqrt{(0.191)^2 + 0.588^2} = 0.618 \text{ rad/s}$$

Problem 2.17

Obtain the eigenvalues of a damped system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\{\ddot{X}\} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\{\dot{x}_1\} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\{X\} = 0$$

Solution to Problem 2.17

Here,

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\tilde{K} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues of the matrix $\tilde{M}^{-1\tilde{K}}$ are

$$\lambda_{1,2=-1.309\pm0.951\ i}$$
$$\lambda_{3,4=-0.191\pm0.588\ i}$$

Hence,

$$\omega_2 = \sqrt{1.309^2 + 0.951^2} = 1.618 \text{ rad/s}$$

$$\omega_1 = \sqrt{(0.191)^2 + 0.588^2} = 0.618 \text{ rad/s}$$

Problem 2.18

A three-degrees-of-freedom undamped system described by the following equations is subjected to an impulse of magnitude I. Determine the resulting motion of the system.

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} I\partial(t) \\ 0 \\ 0 \end{bmatrix}$$

Solution to Problem 2.18

Natural frequencies of the system are given by

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k}{m}}$$

$$\omega_3 = \sqrt{\frac{3k}{m}}$$

The mode shapes are normalized, leading to a modal matrix of

$$P = \frac{1}{\sqrt{6m}} \begin{bmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & -1 \end{bmatrix}$$

Hence, the force vector is

$$G = P^T \cdot F = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ -1 \end{bmatrix} \frac{I}{\sqrt{6m}} \partial(t)$$

Therefore, the differential equations in principal coordinates are

$$\ddot{p}_1 = \frac{I}{\sqrt{3m}}\partial(t)$$

$$\ddot{p}_2 + \frac{k}{m}p_2 = \frac{I}{\sqrt{2m}}\partial(t)$$

$$\ddot{p}_3 + \frac{3k}{m}p_3 = \frac{-I}{\sqrt{6m}}\partial(t)$$

With the initial conditions of $p_i(0) = \dot{p}_i(0) = 0$,

$$p_1 = \frac{I}{\sqrt{3m}}tu(t)$$

$$p_2 = \frac{I}{\sqrt{2m}}\sin\sqrt{\frac{k}{m}}tu(t)$$

$$\ddot{p}_3 = \frac{-I}{\sqrt{6m}}\sin\sqrt{\frac{3k}{m}}tu(t)$$

Hence, $x_1(t)$ coordinates are obtained as

$$x_1 = \left(\frac{1}{\sqrt{3m}}p_1 + \frac{1}{\sqrt{2m}}p_2 - \frac{1}{\sqrt{6m}}p_3\right)$$

$$= \left(\frac{I}{3m}t + \frac{I}{2m}\sin\sqrt{\frac{k}{m}}t + \frac{I}{6m}\sin\sqrt{\frac{3k}{m}}t\right)u(t)$$

$$x_2 = \left(\frac{1}{\sqrt{3}}p_1 + \frac{2}{\sqrt{6m}}p_3\right)$$

$$= \left(\frac{I}{3m}t - \frac{I}{2m}\sin\sqrt{\frac{3k}{m}}t\right)u(t)$$

and

$$x_3 = \frac{1}{\sqrt{3m}}p_1 - \frac{1}{\sqrt{2m}}p_2 - \frac{1}{\sqrt{6m}}p_3$$

$$= \left(\frac{I}{3m}t - \frac{I}{2m}\sin\sqrt{\frac{k}{m}} + \frac{I}{6m}\sin\sqrt{\frac{3k}{m}}\right)u(t)$$

Problem 2.19

By using modal analysis, determine the resulting motion of the system subjected to a constant torque of magnitude T_0 at a time $t = 0$ at the right-hand mass of the system (Figure 2.19).

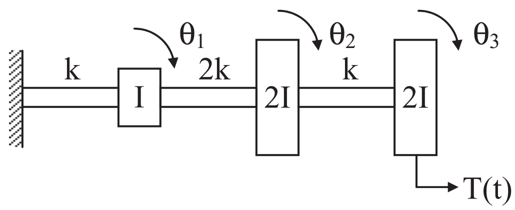


Figure 2.19 Torsional vibrating system.

Solution to Problem 2.19

Here, the equations of motion are written as

$$I\ddot{\theta}_2 = -k(\theta_1) + 2k(\theta_2 - \theta_1)$$
$$2I\ddot{\theta}_2 = -2k(\theta_2 - \theta_1) + k(\theta_3 - \theta_2)$$
$$2I\ddot{\theta}_3 = -k(\theta_3 - \theta_2)$$

or

$$[M]\ddot{\theta} + [K]\theta = 0$$

with

$$[M] = \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 2I \end{bmatrix}$$

and

$$[K] = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix}$$

The natural frequencies are obtained by solving the eigenvalue problem

$$[K - M\omega^2]\phi = 0$$

where $\theta = \phi \sin \omega t$.

Thus, $|M^{-1}K - \lambda I = 0|$, which gives

$$\begin{vmatrix} \left(3\dfrac{k}{I} - \lambda\right) & -2\dfrac{k}{I} & 0 \\ -\dfrac{k}{I} & \left(\dfrac{3}{2}\dfrac{k}{I} - \lambda\right) & \dfrac{-k}{2I} \\ 0 & \dfrac{-k}{2I} & \left(\dfrac{k}{2I} - \lambda\right) \end{vmatrix} = 0$$

that is,

$$-\beta^3 + 5\beta^2 - \frac{9}{2}\beta + \frac{1}{2} = 0$$

or

$$\beta = \lambda \frac{I}{k}$$

The roots of this cubic equation are 0.129, 1, and 3.87. Thus,

$$\omega_1 = 0.359\sqrt{\frac{k}{I}}$$

$$\omega_2 = \sqrt{\frac{k}{I}}$$

$$\omega_3 = 1.97\sqrt{\frac{k}{I}}$$

Correspondingly, the mode shape vectors are

$$\theta_1 = \begin{bmatrix} 0.697 \\ 1 \\ 1.347 \end{bmatrix}$$

$$\theta_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\theta_3 = \begin{bmatrix} -2.298 \\ 1 \\ -0.1484 \end{bmatrix}$$

Normalization of a mode shape vector θ is achieved by dividing every component of the vector by $\left[\theta^T \cdot M \cdot \theta\right]^{1/2}$.

Hence,

$$\theta_1^T \cdot M \cdot \theta_1 = \begin{bmatrix} 0.697 & 1 & 1.347 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 2I \end{bmatrix} \begin{bmatrix} 0.697 \\ 1 \\ 1.347 \end{bmatrix} = (6.115\ I)$$

Likewise, $\theta_2^T \cdot M \cdot \theta_2 = 5\ I$ and $\theta_3^T \cdot M \cdot \theta_3 = 7.325\ I\ \theta_3^T$. The normalized mode shapes are given by

$$\theta_1 = \frac{1}{\sqrt{6.115\ I}} \begin{bmatrix} 0.697 \\ 1 \\ 1.347 \end{bmatrix} = \frac{1}{\sqrt{I}} \begin{bmatrix} 0.2819 \\ 0.4044 \\ 0.5447 \end{bmatrix}$$

$$\theta_2 = \frac{1}{\sqrt{I}} \begin{bmatrix} 0.4472 \\ 0.4472 \\ -0.4472 \end{bmatrix}$$

$$\theta_2 = \frac{1}{\sqrt{I}} \begin{bmatrix} -0.8491 \\ 0.3695 \\ -0.0548 \end{bmatrix}$$

Thus, the modal matrix of the system is

$$\left[P\right] = \frac{1}{\sqrt{I}}\begin{bmatrix} 0.282 & 0.447 & -0.849 \\ 0.404 & 0.447 & 0.370 \\ 0.545 & -0.447 & -0.055 \end{bmatrix}$$

The torque vector is

$$T = \begin{bmatrix} 0 \\ 0 \\ T_0 \end{bmatrix}$$

Hence, the vector $G(t)$ is given by

$$G = \left[P\right]^T T = \frac{1}{\sqrt{I}}\begin{bmatrix} 0.282 & 0.404 & 0.545 \\ 0.447 & 0.447 & -0.447 \\ -0.849 & 0.370 & -0.055 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ T_0 \end{bmatrix} = \frac{1}{\sqrt{I}}\begin{bmatrix} 0.545\,T_0 \\ -0.447\,T_0 \\ -0.055\,T_0 \end{bmatrix}$$

Thus, the differential equations for the principal coordinates become

$$\ddot{p}_1 + 0.129\frac{k}{I}p_1 = 0.545\frac{T_0}{\sqrt{I}}$$

$$\ddot{p}_2 + \frac{k}{I}p_2 = -0.447\frac{T_0}{\sqrt{I}}$$

$$\ddot{p}_3 + 3.88\frac{k}{I}p_3 = -0.055\frac{T_0}{\sqrt{I}}$$

Solutions for the principal coordinates are obtained by solving the individual equations; that is,

$$p_1 = 4.23\frac{\sqrt{I}}{k}T_0\left(1 - \cos 0.359\sqrt{\frac{k}{I}}\,t\right)$$

$$p_2 = -0.447\frac{\sqrt{I}}{k}T_0\left(1 - \cos\sqrt{\frac{k}{I}}\,t\right)$$

$$p_3 = -0.0142\frac{\sqrt{I}}{k}T_0\left(1 - \cos 1.97\sqrt{\frac{k}{I}}\,t\right)$$

The solutions in the original general coordinates are obtained according to

$$\left[\theta\right] = \left[P\right]\left[p\right]$$

Problem 2.20

Obtain the phase plane of a single-degree-of-freedom oscillator, the equation of motion of which is $\ddot{x} + \omega^2 x = 0$.

Solution to Problem 2.20

By substituting $y - \dot{x}$, we can write two first-order equations as

$$\dot{y} = \omega^2 x$$
$$\dot{x} = y$$

On dividing these equations, we get

$$\frac{dy}{dx} = -\frac{\omega^2 x}{y}$$

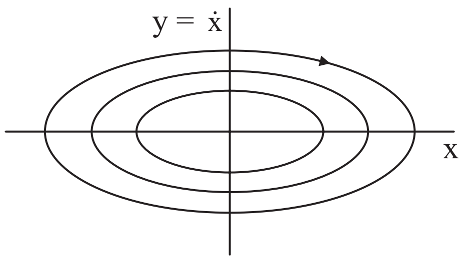


Figure 2.20 Single-degree-of-freedom oscillator.

On separating the variables and integrating, we obtain

$$y^2 + \omega^2 x^2 = C \tag{2.20.1}$$

This is a series of ellipses, the size of which is determined by C. Equation 2.20.1 also corresponds to the conservation of energy, namely,

$$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = C' \tag{2.20.2}$$

Because the singular point is located at $x = y = 0$, the phase plane plot appears as shown in Figure 2.20. If $\frac{y}{\omega}$ is plotted instead of y, the ellipses of Figure 2.20 will be reduced to circles.

Problem 2.21

Derive the equation of motion for the simple pendulum with an attached linear spring system as shown in Figure 2.21. The system is shown in the static equilibrium position.

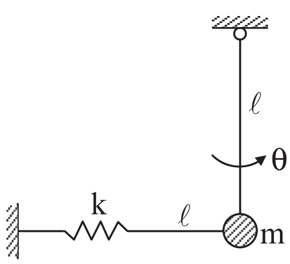


Figure 2.21 Simple pendulum.

Solution to Problem 2.21

We note from Figure 2.21(a),

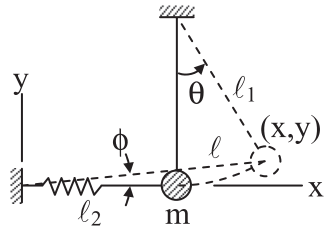


Figure 2.21 (a) Free body diagram.

$$\begin{aligned} x &= \ell_2 + \ell_1 \sin\theta \\ y &= \ell_1(1 - \cos\theta) \end{aligned} \tag{2.21.1}$$

Taking the first derivatives of Eq. 2.21.1, we get

$$\begin{aligned} \dot{x} &= \ell_1\dot{\theta}\cos\theta \\ \dot{y} &= \ell_1\dot{\theta}\sin\theta \end{aligned}$$

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\ell_1^2\dot{\theta}^2$$

The potential energy is

$$u = mg\ell_1(1 - \cos\theta) + u_k(\text{spring})$$

$$u_k(\text{spring}) = \frac{1}{2}k\left(\ell - \ell_2\right)$$

Also,

$$x^2 + y^2 = \ell^2$$

$$\left(\ell_2 + \ell_1\sin\theta\right)^2 + \ell_1^2\left(1 - \cos\theta\right)^2 = \ell^2$$

Hence,

$$\ell\frac{\partial\ell}{\partial\theta} = \ell_1\left(\ell_1\sin\theta + \ell_2\cos\theta\right)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{\ell_1\left(\ell_1\sin\theta + \ell_2\cos\theta\right)}{\sqrt{\left(\ell_2 + \ell_1\sin\theta\right)^2 + \ell_1^2\left(1 - \cos\theta\right)^2}}$$

$$\frac{\partial u_k}{\partial\theta} = k\left(\ell - \ell_2\right)\frac{\partial\ell}{\partial\theta}$$

where $\ell_1\dfrac{\partial\ell}{\partial\theta}$ is defined.

Apply Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial u}{\partial\theta} = 0$$

Hence,

$$m\ell_1^2\ddot{\theta} + mg\ell_1 \sin\theta + k\left(\ell - \ell_2\right)\frac{\partial \ell}{\partial \theta} = 0$$

If θ is small, $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$, and second-order terms are neglected, we have

$$\left(\ell - \ell_2\right) = \ell_1\theta$$

$$\frac{\partial \ell}{\partial \theta} = \ell_1$$

Therefore,

$$m\ell_1^2\ddot{\theta} + \left(mg\ell_1 + k\ell_1^2\right)\theta = 0$$

Problem 2.22

Derive the differential equations of motion for the simple pendulum shown in Figure 2.22.

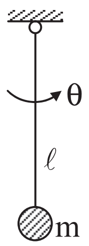


Figure 2.22 Simple pendulum.

Solution to Problem 2.22

The kinetic energy function for the pendulum is given by

$$T = \frac{1}{2}m\left(\ell\dot{\theta}\right)^2 \tag{2.22.1}$$

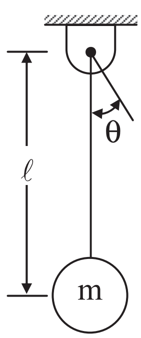


Figure 2.22 (a) Free body diagram.

With the plane of the support as the datum, we have the potential energy

$$V = mg\ell\cos\theta \tag{2.22.2}$$

Applying Lagrange's equation, we have the Lagrangian

$$L = T - V \tag{2.22.3}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{2.22.4}$$

giving

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad (2.22.5)$$

The approximate solution to Eq. 2.22.5 is obtained by its Taylor series expansion. Hence,

$$\ddot{\theta} + \frac{g}{\ell}\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right)$$

Problem 2.23

Derive the differential equations of motion of the system shown in Figure 2.23.

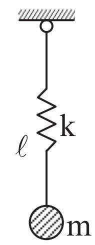


Figure 2.23 Vibrating system.

Solution to Problem 2.23

Let x, the change in length of the spring from its length when the system is in equilibrium with a length ℓ, and θ be the generalized coordinates. The kinetic energy function of the system is given by

$$T = \frac{1}{2}m\left[\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\right] \qquad (2.23.1)$$

The potential energy function of the system is given by

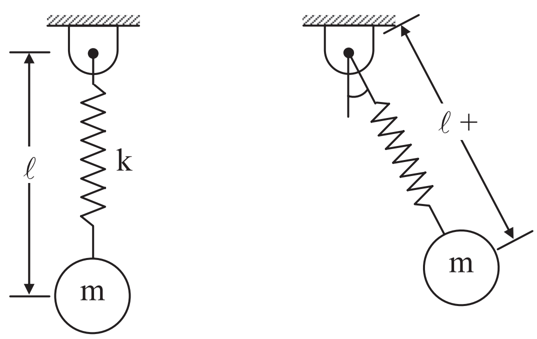
$$V = \frac{1}{2}k\left(x + \frac{mg}{k}\right)^2 - mg(\ell + x)\cos\theta \qquad (2.23.2)$$

Applying Lagrange's equation gives

$$m\ddot{x} + kx - m(\ell + x)\dot{\theta}^2 + mg(1 - \cos\theta) \qquad (2.23.3)$$

and

$$m(\ell + x)^{2\ddot{\theta}} + m(\ell + x)g\sin\theta + 2m(\ell + x)\dot{x}\dot{\theta} = 0 \qquad (2.23.4)$$



(a) (b)

Figure 2.23 Free body diagrams.

If x and θ are assumed to be small, Taylor series expansions can be used for the transcendental functions and only the linear terms are retained. Then, the differential Eqs. 2.23.3 and 2.23.4 become

$$m\ddot{x} + kx = 0 \tag{2.23.5}$$

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \tag{2.23.6}$$

Two uncoupled modes, a spring mode with natural frequency of $\sqrt{k/m}$ and a pendulum mode with a natural frequency of $\sqrt{g/\ell}$, are present. If only the largest nonlinear terms are retained, then the governing differential equations can be written as

$$m\ddot{x} + kx - m\ell\dot{\theta}^2 + \frac{mg}{2}\theta^2 = 0 \tag{2.23.7}$$

$$\ell\ddot{\theta} + g\theta + \frac{g}{\ell}\theta x + 2\dot{x}\dot{\theta} = 0 \tag{2.23.8}$$

Problem 2.24

Duffing's equation for free vibration with $\mu = 0$ is given by

$$\ddot{x} + x + \varepsilon x^3 = 0$$

Determine integral expression for the natural period, assuming $x = x_0$ and $\dot{x} = 0$ when $t = 0$.

Solution to Problem 2.24

Duffing's equation for free vibration with $\mu = 0$ is

$$\ddot{x} + x + \varepsilon x^3 = 0 \tag{2.24.1}$$

Let $\upsilon = \dot{x}$. Then, we have

$$\upsilon\frac{d\upsilon}{dx} + x + \varepsilon x^3 = 0 \tag{2.24.2}$$

Integrating Eq. 2.24.2 with respect to x gives

$$\frac{1}{2}\upsilon^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4 = C \tag{2.24.3}$$

Applying initial conditions and solving for υ gives

$$\upsilon = \pm\sqrt{x_0^2 + \frac{\varepsilon}{2}x_0^4 - x^2 - \frac{\varepsilon}{2}x^4} \tag{2.24.4}$$

Because

$$\upsilon = \frac{dx}{dt}$$

we have

$$Dt = \pm \frac{dx}{\sqrt{x_0^2 + \frac{\varepsilon}{2}x_0^4 - x^2 - \frac{\varepsilon}{2}x^4}} \tag{2.24.5}$$

One-quarter of the period is the time the block returns to $x = 0$ from its initial position. During this time, the velocity is negative. Hence, integrating between x_0 and 0 gives

$$T = 4\int_{x_0}^{0} \frac{dx}{-\sqrt{x_0^2 + \frac{\varepsilon}{2}x_0^4 - x^2 - \frac{\varepsilon}{2}x^4}} \tag{2.24.6}$$

Problem 2.25

Figure 2.25 shows a simple pendulum. If it is assumed that the pendulum is released from initial angle θ_0 with zero angular velocity, and that no friction is present at the pin joint, determine the following:

a. The natural frequency for small angles of oscillation

b. The natural frequency by using elliptic integrals for $\theta_0 = 90°$ and a two-term power series approximation for $\sin\theta$

c. The natural frequency by using the exact form of the differential equation

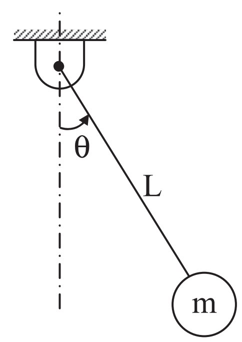


Figure 2.25 Simple pendulum.

Solution to Problem 2.25

a. For small angles of θ, the differential equation of motion can be written as

$$mL^2\ddot{\theta} = -mgL\sin\theta \tag{2.25.1}$$

or

$$\ddot{\theta} + (g/L)\theta = 0 \tag{2.25.2}$$

where $\sin\theta$ is replaced by θ. Hence,

$$\omega_n = \sqrt{g/L}\ \text{rad/s}$$
$$T = \frac{6.28}{\sqrt{g/L}}\ \text{s} \tag{2.25.3}$$

b. When $\sin\theta$ is replaced by $(\theta - \theta^3/6)$, the equation of motion can be written as

$$\ddot{\theta} + (g/L)\theta - (g/6L)\theta^3 = 0 \tag{2.25.4}$$

Let

$$\frac{d\theta}{dt} = t$$

and

$$\frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$$

we get

$$\int_{\omega_0}^{\omega} \omega\, d\omega = -\int_{\theta_0}^{\theta} \frac{g\theta}{L} d\theta + \int_{\theta_0}^{\theta} \frac{g\theta^3}{6L} d\theta \tag{2.25.5}$$

which gives

$$\omega^2 = \frac{g}{L}\left(\theta_0^2 - \theta^2\right) - \frac{g}{12L}\left(\theta_0^4 - \theta^4\right)$$

or

$$\omega = -\sqrt{\frac{g}{L}\left(\theta_0^2 - \theta^2\right)}\left[\sqrt{1 + \frac{1}{12}\left(\theta_0^2 + \theta^2\right)}\right] = \frac{d\theta}{dt} \tag{2.25.6}$$

Let $\theta = \theta_0 \cos\phi$; then,

$$\sqrt{\theta_0^2 - \theta^2} = \theta_0 \sin\phi$$

$$\theta_0^2 + \theta^2 = \theta_0^2\left(1 + \cos^2\phi\right)$$

$$d\theta = -\theta_0 \sin\phi\, d\phi$$

$$\omega = \frac{d\theta}{dt} = -\theta_0 \sin\phi\, \frac{d\phi}{dt}$$

By substituting these values into the expression for ω and integrating, we get

$$t = \frac{1}{\sqrt{g/L}} \int_0^{\theta} \frac{d\phi}{\sqrt{1 + \frac{1}{12}\theta_0^2\left(1 + \cos^2\phi\right)}} \tag{2.25.7}$$

Let

$$k_1^2 = -\frac{g\theta_0^2}{12L}\left(g/L - g\theta_0^2/L\right)$$

Then,

$$\frac{g}{L}\left[1 + \frac{\theta_0^2}{12}\left(1 + \cos^2\phi\right)\right] = \left[g/L + (g/6L)(\theta_0)^2\right]\left(1 - k_1^2 \sin^2\phi\right) \tag{2.25.8}$$

and

$$t = \frac{1}{\sqrt{g/L - g\theta_0^2/6L}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 + k_1^2 \sin^2 \phi}} \tag{2.25.9}$$

Now, θ0 = 90 or $\pi/2$, and

$$t = \frac{1}{1.19\sqrt{g/L}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k_1^2 \sin^2 \phi}} \tag{2.25.10}$$

which is an incomplete elliptic integral of the first kind. Hence,

$$T = \frac{4}{1.19\sqrt{g/L}} \int_0^{\phi} \frac{d\theta}{\sqrt{1 - 0.386 \sin^2 \theta}} = \frac{7.32}{\sqrt{g/L}}\ \mathrm{s} \tag{2.25.11}$$

c. Using the exact form of the differential equation of motion also leads to an expression for the oscillation period, which is an elliptic integral of the first kind. Hence,

$$t = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{1 - \frac{2g}{L}\left(\cos\theta_0 - \cos\theta\right)}} \tag{2.25.12}$$

and

$$\begin{aligned} T &= \frac{4}{\sqrt{g/L}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2\left(\theta_0/2\right)\sin^2\theta}} \\ &= \frac{4}{\sqrt{g/L}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - 0.49 \sin^2 \theta}} = \frac{7.41}{\sqrt{g/L}}\ \mathrm{s} \end{aligned} \tag{2.25.13}$$

Problem 2.26

Determine the equations for the cumulative probability and the probability density functions of the sine wave.

Solution to Problem 2.26

$$x = A \sin\theta$$

when

$$x = 0$$

$$P(x) = \frac{1}{2}$$

In other words, half the time x is less than x = 0.

As x increases from x = 0, we add $\frac{2\theta}{2\pi}$ to the probability, and

$$\theta = \sin^{-1}\frac{x}{A}$$

Therefore,

$$P(x) = \frac{1}{2} \pm \sin^{-1}\frac{x}{A}$$

Hence,

$$P(x) = \frac{dP(x)}{dx} = \frac{1}{\pi}\frac{d}{dx}\left(\sin^{-1}\frac{x}{A}\right) = \frac{1}{\pi\sqrt{A^2 - x^2}}$$

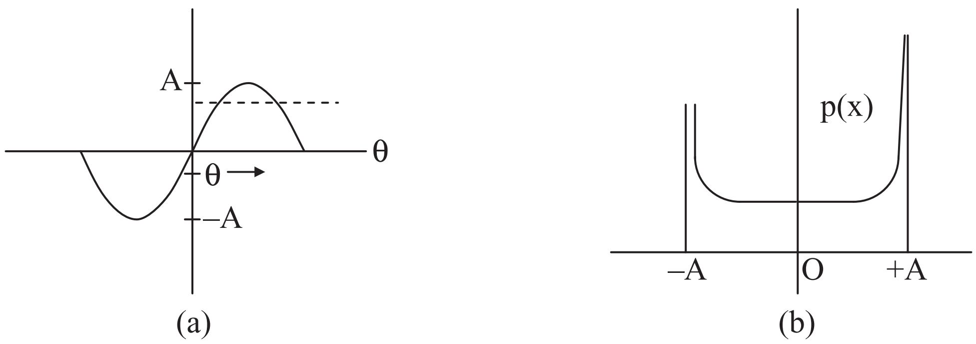


Figure 2.26 (a) Sine wave and (b) cumulative probability density function.

Problem 2.27

The probability density function of a random variable x is given by

$$p(x) = \begin{cases} kx^2 & 0 \le x \\ 0 & \text{elsewhere} \end{cases}$$

where k is a constant. Determine the mean, the standard deviation, and the mean square value of x.

Solution to Problem 2.27

As stated, the probability density function of a random variable x is given as

$$p(x) = \begin{cases} kx^2 & 0 \le x \\ 0 & \text{elsewhere} \end{cases} \tag{2.27.1}$$

where k is a constant.

The value of k in Eq. 2.27.1 can be obtained by normalizing the probability density function as

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^5 kx^2\,dx = 1$$

or

$$k\left(\frac{x^3}{3}\right)_0^5 = 1$$

Hence,

$$k = \frac{3}{125} \tag{2.27.2}$$

a. The mean value of x is given by

$$\ddot{x}\int_0^5 p(x)\,x\,dx = k\left(\frac{x^4}{4}\right)_0^5 = 3.75 \tag{2.27.3}$$

The standard deviation of x is given by

$$\sigma_x^2 = \int_0^5 (x - \ddot{x})^2 p(x)\,dx = \int_0^5 (x^2 + \ddot{x}^2)\,p(x)\,dx$$

$$= \int_0^5 kx^4 dx - (\ddot{x})^2 = k\left(\frac{x^5}{5}\right)_0^5 - (\ddot{x})^2$$

$$= k\left(\frac{3125}{5}\right) - (3.75)^2 - 0.9375$$

Hence,

$$\sigma_x = 0.9682 \tag{2.27.4}$$

The mean square of x is

$$\overline{x}^2 = k\left(\frac{3125}{5}\right) = 15 \tag{2.27.5}$$

Problem 2.28

Determine the following:

a. The temporal mean value and autocorrelation function of the function $x(t)$ given by

$$x(t) = \begin{cases} 0 & -\dfrac{T}{2} < t < 0 \\ \dfrac{2A}{T}t & 0 < t < \dfrac{T}{2} \end{cases}$$

by using the probability density function of $x(t)$.

b. The mean square value, the variance, and the standard deviation of the function $x(t)$.

Solution to Problem 2.28

a. The mean value is given by

$$\mu_x = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} \frac{2A}{T}t\,dt = \frac{A}{4} \tag{2.28.1}$$

To obtain the autocorrelation function, we distinguish between the time shifts $0 < \tau < T/2$ and $T/2 < \tau < T$, as shown in Figures 2.28(a) and (b), respectively. We obtain for $0 < \tau < T/2$ from Figure 2.28(a),

$$\begin{aligned} R_x(\tau) &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t+\tau)\,dt = \frac{1}{T}\int_{0}^{(T/2)-\tau} \frac{2A}{T}t\,\frac{2A}{T}(t+\tau)\,dt \\ &= \frac{A^2}{6}\left[1 - 3\frac{\tau}{T} + 4\left(\frac{\tau}{T}\right)3\right] \quad 0 < \tau < \frac{T}{2} \end{aligned} \tag{2.28.2}$$

where the limits of integration are defined by the overlapping portions of $x(t)$ and $x(t+\tau)$ in the shaded area in Figure 2.28(a).

From Figure 2.28(b), we obtain for $T/2 < \tau < T$

$$\begin{aligned} R_x(T) &= \frac{1}{T}\int_{T-\tau}^{T/2} \frac{2A}{T}t\,\frac{2A}{T}\left[t - (T-\tau)\right]dt \\ &= \frac{A^2}{6}\left[1 - \frac{3}{T}(T-\tau) + \frac{4}{T^3}(T-\tau)^3\right] \qquad \frac{T}{2} < \tau < T \end{aligned} \tag{2.28.3}$$

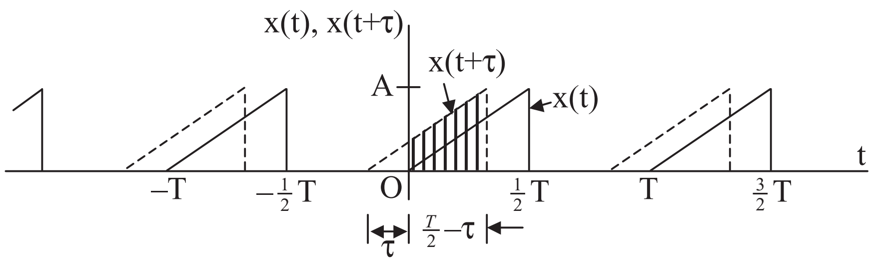


Figure 2.28 (a) Function x(t).

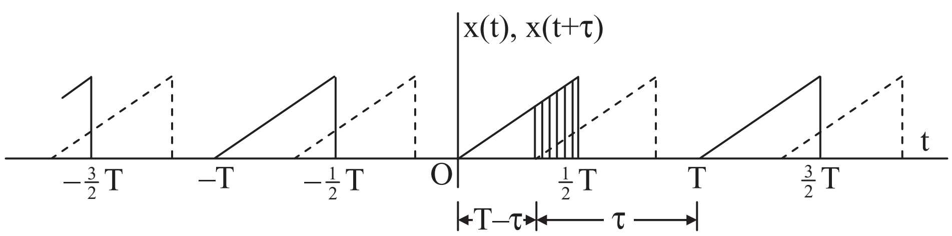


Figure 2.28 (b) Function x(t).

From Figure 2.28, we conclude that the autocorrelation function $R_x(\tau)$ must be periodic in τ with period T. Hence, from Eqs. 2.28.2 and 2.28.3, and the fact that $R_x(\tau)$ is periodic, we can obtain the autocorrelation function plot as shown in Figure 2.28(c).

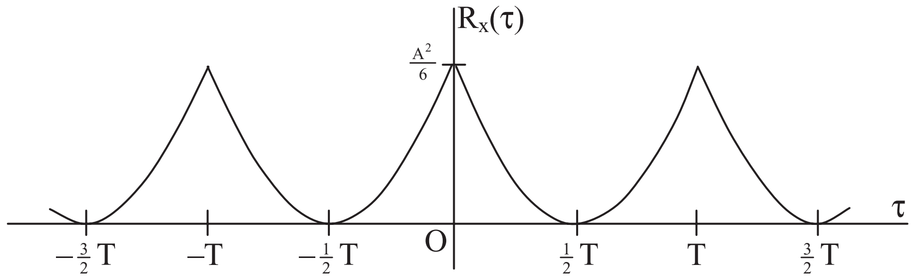


Figure 2.28 (c) Auto-correlation function plot.

b. We have

$$R_x(k,\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_k(t)x_k(t+\tau)dt \tag{2.28.4}$$

$$=$$

$$\psi_x^2 = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt \tag{2.28.5}$$

Upon comparing Eqs. 2.28.4 and 2.28.5, we conclude that $\psi_x^2 = R_x(0)$, or the mean square value is equal to the autocorrelation function evaluated at $\tau = 0$. Hence, from Eq. 2.28.3, the mean square value is given by

$$\psi_x^2 = R_x(0) = \frac{A^2}{6} \tag{2.28.6}$$

Introducing the above and Eq. 2.28.1 into the variance equation, we obtain the variance

$$\sigma_x^2 = \psi_x^2 - \mu_x^2 = \frac{A^2}{6} - \left(\frac{A}{4}\right)^2 = \frac{5}{48}A^2 \tag{2.28.7}$$

The standard deviation is given by

$$\sigma_x = \sqrt{\frac{5}{48}}A \tag{2.28.8}$$

Problem 2.29

Determine the complex Fourier series expansion of the function shown in Figure 2.29(a).

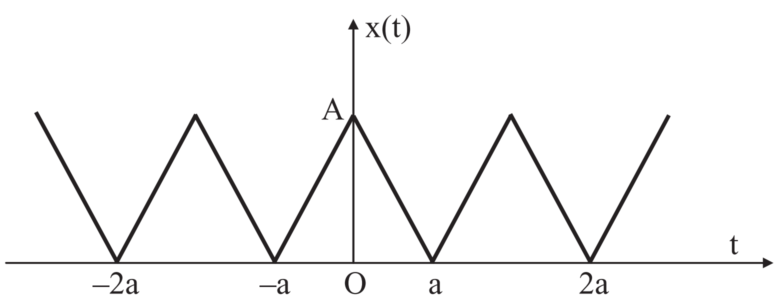


Figure 2.29 (a) Function $x(t)$.

Solution to Problem 2.29

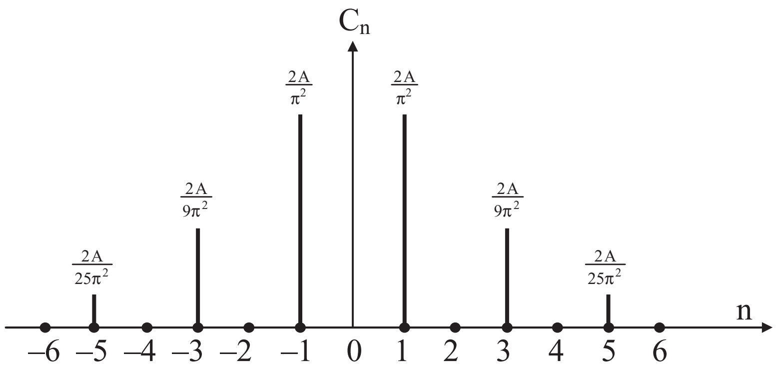


Figure 2.29 (b) Function $x(t)$.

The given function can be written as

$$x(t) = \begin{cases} A\left(1 + \frac{t}{a}\right), & -\frac{\tau}{2} \le t \le 0 \\ A\left(1 - \frac{t}{a}\right), & 0 \le t \le \frac{\tau}{2} \end{cases} \tag{2.29.1}$$

where the period (τ) and the fundamental frequency (ω_0) are given by

$$\tau = 2a \qquad \text{and} \qquad \omega_0 = \frac{2\pi}{\tau} = \frac{\pi}{a} \tag{2.29.2}$$

The Fourier coefficients are

$$\begin{aligned} c_n &= \frac{1}{\tau}\int_{-\pi/2}^{\pi/2} x(t)e^{-in\omega_0 t}\omega t \\ &= \frac{1}{\tau}\left[\int_{-\tau/2}^{0} A\left(1 + \frac{t}{a}\right)e^{-in\omega_0 t}\,dt + \int_{0}^{\tau/2} A\left(1 - \frac{t}{a}\right)e^{-in\omega_0 t}\,dt\right] \end{aligned} \tag{2.29.3}$$

By using the relation,

$$\int te^{kt}dt\,\frac{e^{kt}}{k^2}(kt - 1) \tag{2.29.4}$$

c_n can be evaluated as

$$\begin{aligned} c_n = \frac{1}{\tau}\Bigg[&\frac{A}{-in\omega_0}e^{-in\omega_0 t}\Bigg|_{-\tau/2}^{0} + \frac{A}{a}\left\{\frac{e^{-in\omega_0 t}}{(-in\omega_0)^2}\left[-in\omega_0 t - 1\right]\right\}\Bigg|_{-\tau/2}^{0} \\ &+ \frac{A}{-in\omega_0}e^{-in\omega_0 t}\Bigg|_{0}^{\tau/2} - \frac{A}{a}\left\{\frac{e^{-in\omega_0 t}}{(-in\omega_0)^2}\left[-in\omega_0 t - 1\right]\right\}\Bigg|_{0}^{\tau/2}\Bigg] \end{aligned} \tag{2.29.5}$$

Equation 2.29.5 can be reduced to

$$c_n = \frac{1}{\tau}\left[\frac{A}{-in\omega_0}e^{in\pi} + \frac{2A}{a}\frac{1}{n^2\omega_0^2} - \frac{A}{in\omega_0}e^{-in\pi} - \frac{A}{a}\frac{1}{n^2\omega_0^2}e^{in\pi} - \frac{A}{a}\frac{1}{n^2\omega_0^2}e^{-in\pi} \right.$$
$$\left. + \frac{A}{a}\frac{1}{n^2\omega_0^2}(in\,\pi)e^{in\pi} - \frac{A}{a}\frac{1}{n^2\omega_0^2}(in\,\pi)e^{-in\pi}\right] \qquad (2.29.6)$$

Noting that

$$e^{in\pi} \quad \text{or} \quad e^{-in\pi}\begin{cases} 1, & n = 0 \\ -1, & n = 1,3,5,\ldots \\ 1, & n = 2,4,6,\ldots \end{cases} \qquad (2.29.7)$$

Equation 2.29.6 can be simplified to

$$c_n = \begin{cases} 0, & n = 0 \\ \left(\dfrac{4A}{a\tau n^2\omega_0^2}\right) = \dfrac{2A}{n^2\omega^2} & n = 1,3,5,\ldots \\ 0, & n = 2,4,6,\ldots \end{cases} \qquad (2.29.8)$$

The frequency spectrum is shown in Figure 2.32(b).

Problem 2.30

Show that the power spectral density spectrum of the sine wave

$$f(t) = A\sin\left(\frac{2\pi}{T}\right)t$$

is given by

$$S_f(\omega) = \frac{\pi A^2}{2}\left[\delta\left(\omega + \frac{2\pi}{T}\right) + \delta\left(\omega - \frac{2\pi}{T}\right)\right]$$

where $\delta\left(\omega + \frac{2\pi}{T}\right)$ and $\delta\left(\omega - \frac{2\pi}{T}\right)$ are Dirac delta functions acting at $\omega = \frac{-2\pi}{T}$ and $\omega = \frac{2\pi}{T}$, respectively.

Solution to Problem 2.30

$$R_f(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} A^2 \sin\frac{2}{T}t\sin\frac{2\pi}{T}(t+\tau)dt$$

$$= \frac{1}{2}A^2\cos\frac{2\pi}{T}\tau = \frac{A^2}{4}\left[e^{i\frac{2\pi}{T}\tau} + e^{-i\frac{2\pi}{T}\tau}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)e^{i\omega\tau}\,d\omega$$

Hence,

$$S_f(\omega) = \frac{1}{2}\pi A^2\left[\delta\left(\omega + \frac{2\pi}{T}\right) + \delta\left(\omega - \frac{2\pi}{T}\right)\right]$$

Problem 2.31

The autocorrelation function of a stationary random process $x(t)$ is given by

$$R_x(\tau) = ae^{-b|\tau|}$$

where a and b are constants. Determine the power spectral density of $x(t)$.

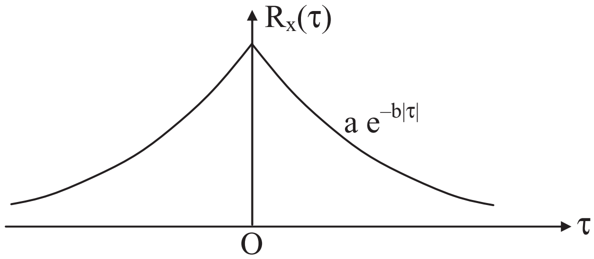


Figure 2.31 Auto-correlation function.

Solution to Problem 2.31

$$R_x(\tau) = ae^{-b|\tau|}$$

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{0} ae^{b\tau} - e^{-i\omega\tau}\,d\tau$$

$$= \left(\frac{a}{2\pi}\right)\frac{1}{-(i\omega - b)}\left[e^{-(i\omega+b)\tau}\right]_{-\infty}^{0} + \left(\frac{a}{2\pi}\right)\frac{1}{-(i\omega + b)}\left[e^{-(i\omega+b)}\right]_{0}^{\infty}$$

$$= \frac{-a}{2\pi(i\omega - b)} + \frac{a}{2\pi(i\omega + b)} = \frac{ab}{\pi(b^2 + \omega^2)}$$

Problem 2.32

Determine the Fourier transform of the function shown in Figure 2.32, and plot the corresponding spectrum.

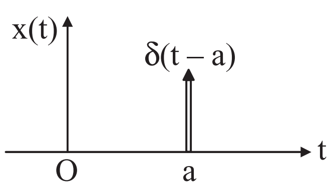


Figure 2.32 Function $x(t)$.

Solution to Problem 2.32

$$x(t) = \delta(t - a)$$

Hence,

$$X(\omega) = \int_{-\infty}^{\infty}\delta(t - a)e^{-i\omega t}\,dt = e^{-i\omega a} = \cos\omega a - i\sin\omega a$$

Note that, by definition, the Dirac delta function is zero everywhere except at t = a. Therefore, at t = a, we have

$$e^{-i\omega t} = e^{-i\omega a}$$

Problem 2.33

Determine the Fourier transform of the function shown in Figure 2.33, and plot the corresponding spectrum.

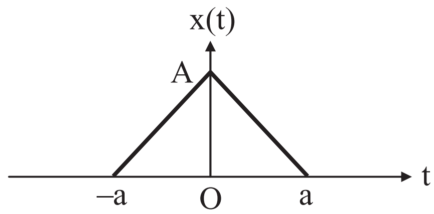


Figure 2.33 Function $x(t)$.

Solution to Problem 2.33

The triangular pulse can be written as

$$x(t) = \begin{cases} A\left(1 - \dfrac{|t|}{a}\right), & |t| \le a \\ 0, & \text{otherwise} \end{cases} \tag{2.33.1}$$

The Fourier transform of $x(t)$ can be written as

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} A\left(1 - \frac{|t|}{a}\right) e^{-i\omega t}\, dt \\ &= \int_{-\infty}^{0} A\left(1 + \frac{t}{a}\right) e^{-i\omega t}\, dt + \int_{0}^{\infty} A\left(1 - \frac{t}{a}\right) e^{-i\omega t}\, dt \end{aligned} \tag{2.33.2}$$

Because $x(t) = 0$ for $|t| > 0$, Eq. 2.33.2 can be written as

$$\begin{aligned} X(\omega) &= \int_{-a}^{0} A\left(1 + \frac{t}{a}\right) e^{-i\omega t}\, dt + \int_{0}^{a} A\left(1 - \frac{t}{a}\right) e^{-i\omega t}\, dt \\ &= \left(\frac{A}{-i\omega}\right) e^{-i\omega t}\Big|_{-a}^{0} + \frac{A}{a}\left\{\frac{e^{-i\omega t}}{(-i\omega)^2}[-i\omega t - 1]\right\}\Bigg|_{-a}^{0} \\ &\quad + \left(\frac{A}{-i\omega}\right) e^{-i\omega t}\Big|_{0}^{a} - \frac{A}{a}\left\{\frac{e^{-i\omega t}}{(-i\omega)^2}[-i\omega t - 1]\right\}\Bigg|_{0}^{a} \end{aligned} \tag{2.33.3}$$

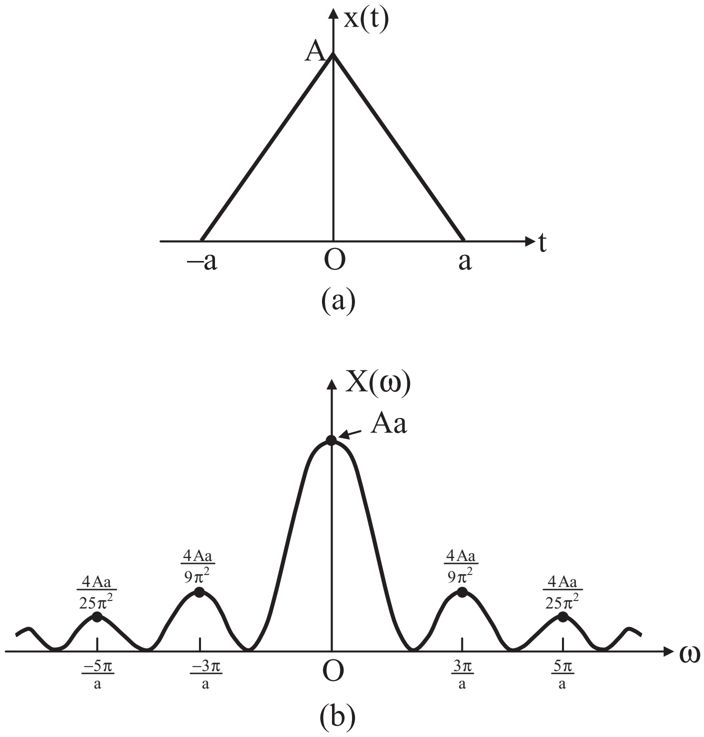


Figure 2.33 (a) Function $x(t)$, and (b) spectrum.

Equation 2.33.3 can be further simplified to obtain

$$\begin{aligned} X(\omega) &= \frac{2A}{a\omega^2} + e^{i\omega a}\left(-\frac{A}{a\omega^2}\right) + e^{-i\omega a}\left(-\frac{A}{a\omega^2}\right) \\ &= \frac{2A}{a\omega^2} - \frac{A}{a\omega^2}(\cos\omega a + i\sin\omega a) - \frac{A}{a\omega^2}(\cos\omega a - i\sin\omega a) \\ &= \frac{2A}{a\omega^2}(1 - \cos\omega a) = \frac{4A}{a\omega^2}\sin^2\left(\frac{\omega a}{2}\right) \end{aligned} \tag{2.33.4}$$

Equation 2.33.4 is shown in Figure 2.33(b).

Problem 2.34

Express the autocorrelation in terms of the Fourier transform.

Solution to Problem 2.34

$$x(t+\tau) = \int_{-\infty}^{\infty} X(f)e^{i2\pi f(t+\tau)}\,df \tag{2.34.1}$$

By substituting Eq. 2.34.1 into the expression for the autocorrelation, we get

$$\begin{aligned} R(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} X(f)e^{i2\pi ft}e^{i2\pi f\tau}\,df\,dt \\ &= \int_{-\infty}^{\infty}\frac{1}{T}\left[\int_{-\infty}^{\infty} x(t)e^{i2\pi ft}\,dt\right]X(f)e^{i2\pi f\tau}\,df \\ &= \int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{T}X*(f)X(f)\right]e^{i2\pi f\tau}\,df \end{aligned} \tag{2.34.2}$$

or

$$R(\tau) = \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau}\, df \qquad (2.34.3)$$

The inverse of Eq. 2.34.3 is also available from the Fourier transform as

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau}\, d\tau \qquad (2.34.4)$$

Because $R(\tau)$ is symmetric about $\tau = 0$, Eq. 2.34.4 can also be written as

$$S(f) = \int_{-\infty}^{\infty} R(\tau) \cos 2\pi f\tau\, d\tau \qquad (2.34.5)$$

These are the *Wiener-Khintchine* equations, and they state that the spectral density function is the FT of the autocorrelation function.

We also can define the cross-correlation between two quantities $x(t)$ and $y(t)$ as

$$\begin{aligned} R_{xy}(\tau) &= \langle x(t) y(t+\tau) \rangle \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) y(t+\tau) dt \\ &= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} X^*(f) Y(f) e^{i2\pi f\tau}\, df \end{aligned}$$

or

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f) e^{i2\pi f\tau}\, df \qquad (2.34.6)$$

where the cross-spectral density is defined as

$$S_{xy}(f) = \lim_{T\to\infty} \frac{1}{T} X(f) Y^*(f) = S_{xy}^*(f) = S_{xy}(-f) \qquad (2.34.7)$$

The inverse of Eq. 2.34.7 from the Fourier transform is given by

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i2\pi f\tau}\, d\tau \qquad (2.34.8)$$

Unlike the autocorrelation, the cross-correlation and cross-spectral density functions are, in general, not even functions. The limits $-\infty$ to $+\infty$ are retained.

Problem 2.35

Show that the frequency response function $H(\omega)$ is the Fourier transform of the impulse response function $h(t)$.

Solution to Problem 2.35

From the convolution integral, the response equation in terms of the impulse response function can be written as

$$x(t) = \int_{-\infty}^{t} f(\xi) h(t - \xi) d\xi \tag{2.35.1}$$

where the lower limit has been extended to $-\infty$ to account for all past excitations. By letting $\tau = (t - \xi)$, Eq. 2.35.1 becomes

$$x(t) = \int_{0}^{\infty} f(t - \tau) h(\tau) d\tau \tag{2.35.2}$$

For a harmonic excitation $f(t) = e^{i\omega t}$, Eq. 2.35.2 becomes

$$x(t) = \int_{0}^{\infty} e^{i\omega(t-\tau)} h(\tau)\, d\tau = e^{i\omega t} \int_{0}^{\infty} h(\tau) e^{-i\omega\tau}\, d\tau \tag{2.35.3}$$

Because the steady-state output for the input $y(t) = e^{i\omega t}$ is $x = H(\omega) e^{i\omega t}$, the frequency response function is given by

$$H(\omega) = \int_{0}^{\infty} h(\tau) e^{-i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} h(\tau) e^{-i\omega\tau}\, d\tau \tag{2.35.4}$$

which is the FT of the impulse response function $h(t)$. The lower limit in Eq. 2.35.4 has been changed from 0 to $-\infty$ because $h(t) = 0$ for negative t.

Chapter 3

Tire Dynamics

Problem 3.1

The mass of a vehicle is m = 1820 kg, the distance between the center of the mass and the front axle is $l_a = 1.463$ m, and the distance to the rear axle is $l_b = 1.585$ m. Assume that the rolling resistance coefficient between the road and the tire is $f_0 = 0.0165$, when the vehicle is running at a speed of v = 36 km/h and a steering angle of $\delta_0 = 20°$, and the slip angles of the front and rear tires are both 4°. Calculate the rolling resistance of the tire.

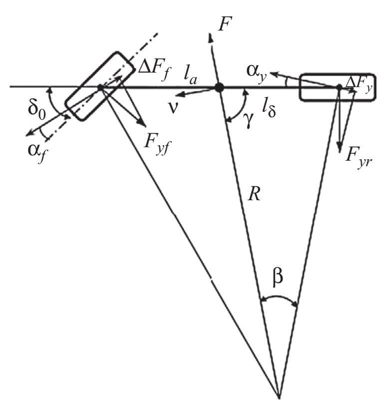


Figure 3.1 Motorcycle model.

Solution to Problem 3.1

According to the relationships of trigonometric function,

$$\frac{R'}{\sin\left(90° - \delta_0 + \alpha_f\right)} = \frac{l_a + l_b}{\sin\left(\delta_0 - \alpha_f + \alpha_r\right)} \tag{3.1.1}$$

$$\begin{aligned} R' &= \frac{(l_a + l_b)\sin(90° - \delta_0 + \alpha_f)}{\sin(\delta_0 - \alpha_f + \alpha_r)} \\ &= \frac{(1.463 + 1.585) \times \sin(90° - 20° + 4°)}{\sin(20° - 4° + 4°)} \\ &= 8.567 \text{ m} \end{aligned} \tag{3.1.2}$$

$$\begin{aligned} R &= \sqrt{l_b^2 + R'^2 - 2l_b \cdot R' \cos(90° - \alpha_r)} \\ &= \sqrt{1.585^2 + 8.567^2 - 2 \times 1.585 \times 8.567 \times \cos(90° - 4°)} \\ &= 8.603 \text{ m} \end{aligned} \tag{3.1.3}$$

$$\beta = \arcsin\left(\frac{l_b \cos\alpha_r}{R}\right) = \arcsin\left(\frac{1.585 \times \cos 4°}{8.603}\right) = 10.6° \tag{3.1.4}$$

$$\gamma = 90° - \beta + \alpha_r = 90° - 10.6° + 4° = 83.4° \tag{3.1.5}$$

The cornering force of the front wheel is

$$\begin{aligned} F_{yf} &= m\frac{v^2}{R} \cdot \frac{\sin(\beta - \alpha_r)}{\sin\delta_0} \\ &= 1820 \times \frac{10^2}{8.603} \cdot \frac{\sin(10.6° - 4°)}{\sin 20°} \\ &= 7109 \text{ N} \end{aligned} \tag{3.1.6}$$

The cornering force of the rear wheel is

$$\begin{aligned} F_{yr} &= m\frac{v^2}{R} \cdot \left[\cos(\beta - \alpha_r) - \sin(\beta - \alpha_r) \cdot \operatorname{ctan}\delta_0\right] \\ &= 1820 \times \frac{10^2}{8.603} \cdot \left[\cos(10.6° - 4°) - \sin(10.6° - 4°) \cdot \operatorname{ctan}20°\right] \\ &= 14{,}335 \text{ N} \end{aligned} \tag{3.1.7}$$

The additional resistance respectively applied on the front and rear wheels is

$$\begin{cases} \Delta F_f = \sin\alpha_f \cdot F_{yf} = \sin 4° \times 7109 = 496 \text{ N} \\ \Delta F_r = \sin\alpha_r \cdot F_{yr} = \sin 4° \times 14{,}335 = 1000 \text{ N} \end{cases} \tag{3.1.8}$$

The additional resistance coefficient under the condition of vehicle steering is

$$\Delta f = \frac{\Delta F_f + \Delta F_r}{mg} = \frac{496 + 1000}{1820 \times 9.8} = 0.0839 \tag{3.1.9}$$

The rolling resistance coefficient during a turn is

$$f_R = f_0 + \Delta f = 0.0165 + 0.0839 = 0.1004 \tag{3.1.10}$$

The rolling resistance of the tire is

$$f = mgf_R = 1820 \times 9.8 \times 0.1004 = 1791\ \text{N} \tag{3.1.11}$$

Problem 3.2

Based on the brush model, the load of the tire is 4500 N, the contact patch length of the tires is 8 cm, the slip ratio is 0.15, and the longitudinal stiffness of the tire is $c_{ex} = 1300\ \text{N/cm}^2$. Calculate the longitudinal force under this slip ratio.

Solution to Problem 3.2

Assume that the vertical forces distribute as a quadratic function. Therefore,

$$F_z = \int_{-4}^{4} \lambda\left(4^2 - x^2\right)dx = 4500\ \text{N} \tag{3.2.1}$$

We get $\lambda = 52.7$. Thus,

$$F_{ez}\left(x\right) = 52.7 \times \left(16 - x^2\right) \tag{3.2.2}$$

The length of the sliding region is

$$l_a = c_{ex}\sigma/(\mu\lambda) = 1300 \times 0.15/\left(1.15 \times 0.8 \times 52.7\right) = 4\ \text{cm} \tag{3.3.3}$$

Therefore,

$$F_x = \int_{-4}^{0} \mu\lambda\left(4^2 - x^2\right)dx + \int_{0}^{4} c_{ex}\sigma\left(x - 4\right)dx = 3450.83\ \text{N} \tag{3.3.4}$$

Problem 3.3

The mass of one car is 1600 kg. If it is a front-engine front-wheel-drive (FF) car, then the load of the front shaft is 61% of the whole car. If it is a front-engine rear-wheel-drive (FR) car, then the load of the front shaft is 55% of the whole car. The coefficient of road adhesion is $\varphi = 0.7$.

a. Calculate the driving force of the FF car.

b. Calculate the driving force of the FR car.

Solution to Problem 3.3

a. For the FF car,

$$G = mg = 1600\ \text{kg} \times 9.8\ \text{N/kg} = 15{,}680\ \text{N} \tag{3.3.1}$$

$$F_z = 0.61\ G = 0.61 \times 15{,}680\ \text{N} = 9564.8\ \text{N} \tag{3.3.2}$$

$$F_x = F_z j = 9564.8\ \text{N} \times 0.7 = 6695.36\ \text{N} \tag{3.3.3}$$

b. For the FR car,

$$G = mg = 1600 \text{ kg} \times 9.8 \text{ N/kg} = 15,680 \text{ N} \tag{3.3.4}$$

$$F_z = (1 - 0.55)G = (1 - 0.55) \times 15,680 \text{ N} = 7056 \text{ N} \tag{3.3.5}$$

$$F_x = F_z\varphi = 7056 \text{ N} \times 0.7 = 4939.2 \text{ N} \tag{3.3.6}$$

Problem 3.4

The load of the tire is $F_z = 4500$ N. The coefficient of road adhesion is $\varphi_p = 1.0$ and $\varphi_s = 0.7$. When the slip ratio is $s = 0.04$, the longitudinal force is $F_x = 1000$ N. When $s = 0.2$, then F_x reaches the maximum value. On the basis of the H.B. Pacejka magic formula model, analyze the relationships between the longitudinal force and the slip ratio.

Solution to Problem 3.4

If the MATLAB program is used,

```
fz = 4500, mp = 1.0, ms = 0.7, s1 = 0.04, f1 = 1000, sp = 0.2;
fp = fz*mp;
fs = fz*ms;
D = fp;
C = 1+(1-2*asin(fs/D)/pi);
B = (f1/s1)/(C*D);
E = (sp*B-tan(pi/(2*C)))/(B*sp-atan(B*sp));
x = 0:0.01:1;
y = D*sin(C*atan(B*x-E*(B*x-atan(B*x))));
plot(x,y)
```

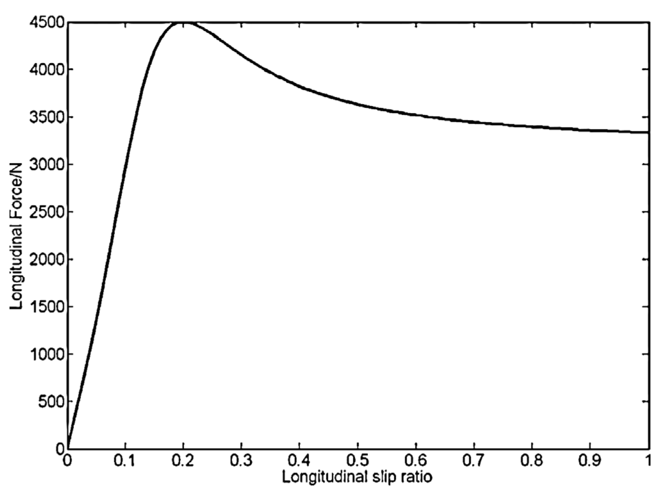


Figure 3.4 Relationships between the longitudinal force and the slip ratio.

PROBLEM 3.5

The distribution function of tire load is as follows:

$$\eta(u) = \begin{cases} 1 & 0 < u < 2 \\ 0 & u \le 0 \quad \text{or} \quad u \ge 2 \end{cases}$$

Calculate the tire-cornering characteristic under the simple distribution load of the tire ($\frac{F_y}{\varphi \cdot F_z} \sim \phi_y$ and $\frac{T_z}{\varphi \cdot F_z \cdot a} \sim \phi_y$).

Solution to Problem 3.5

When $\mu' < 2$, then,

$$\phi_y = \frac{\eta(\mu')}{\mu'} = \frac{1}{\mu'}$$

$$\mu' = \frac{1}{\phi_y}$$

$$m_0(\mu') = \int_0^{\mu'} \eta(\mu)d\mu = \int_0^{\mu'} 1\, d\mu = \mu'$$

Therefore,

$$\frac{F_y}{\varphi \cdot F_z} = \frac{\phi_y}{4} \times \mu'^2 + 1 - \frac{m_0(\mu')}{2} = \frac{\mu'}{4} + 1 - \frac{\mu'}{2} = 1 - \frac{1}{4 \cdot \phi_y} \qquad (3.5.1)$$

When $\mu' = 2$,

$$\frac{F_y}{\varphi \cdot F_z} = \phi_y \qquad (3.5.2)$$

When $\mu' > 2$,

$$\frac{F_y}{\varphi \cdot F_z} \sim \phi_y \qquad (3.5.3)$$

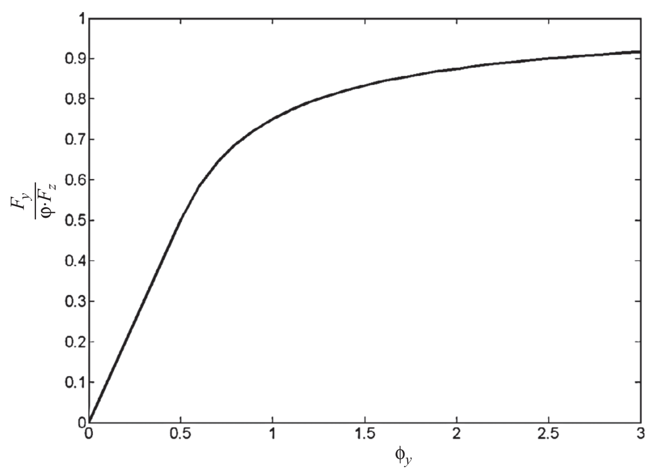


Figure 3.5 (a) Relationships between the lateral force and the relative slip ratio.

The aligning torque is

$$\mu' = \frac{1}{\phi_y}, \quad m_0(\mu') = \int_0^{\mu'} \eta(\mu)\,d\mu = \mu' = \frac{1}{\phi_y} \tag{3.5.4}$$

$$m_1(\mu') = \int_0^{\mu'} \mu \cdot \eta(\mu)\,d\mu = \frac{1}{2}\mu'^2 = \frac{1}{2\phi_y^2} \tag{3.5.5}$$

Therefore, when $\mu' < 2$,

$$\begin{aligned} \frac{T_z}{\varphi \cdot F_z \cdot a} &= \frac{\mu'^2}{2}\left(\frac{\mu'}{3} - \frac{1}{2}\right) \times \phi_y - \theta + \frac{m_0(\mu') - m_1(\mu')}{2} \\ &= \frac{1}{4\phi_y}\left(1 - \frac{1}{3\phi_y}\right) \end{aligned} \tag{3.5.6}$$

When $\mu' \geq 2$,

$$\frac{T_z}{\varphi \cdot F_z \cdot a} = \frac{\phi_y}{3} \tag{3.5.7}$$

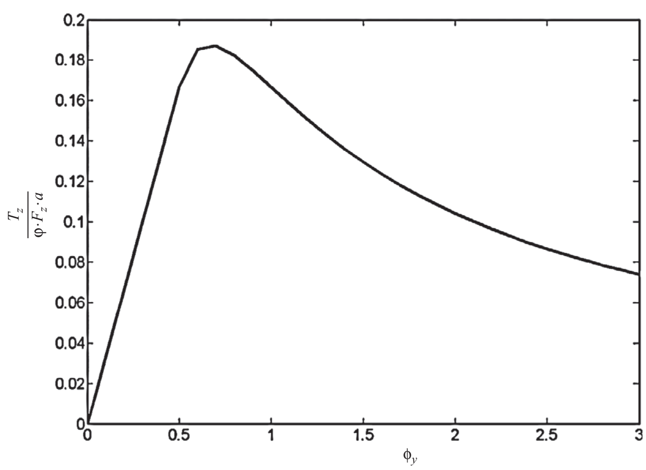


Figure 3.5 (b) Relationships between the aligning torque and the relative slip ratio.

Problem 3.6

What is slip ratio?

Draw a curve to show the relationship between the coefficient of road adhesion and the longitudinal slip ratio. Then describe the characteristics of the curve, and analyze the effects of different factors on the coefficient of road adhesion.

Solution to Problem 3.6

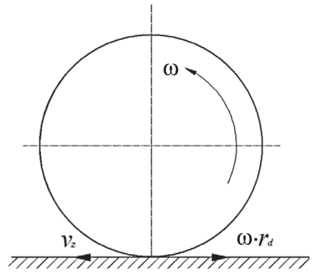


Figure 3.6 (a) Tire model.

The braking maneuver longitudinal skid is s_b.

$$s_b = \frac{v_x - \omega \cdot r_d}{v_x} \times 100\% \tag{3.6.1}$$

The tractive maneuver longitudinal slip is s_a.

$$s_a = \frac{\omega \cdot r_d - v_x}{\omega \cdot r_d} \times 100\% \tag{3.6.2}$$

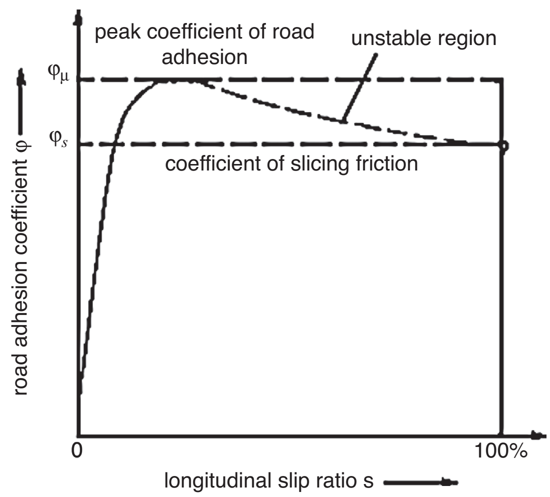


Figure 3.6 (b) Free body diagram.

The $\varphi \sim s$ curve has the following characteristics:

1. If $|s| = 0 \sim 15\%$, the value of φ increases linearly with s.
2. If $|s| = 15\% \sim 30\%$, the value of φ reaches the maximum and is defined as the peak coefficient of road adhesion.
3. If $|s| = 30\% \sim 100\%$, the value of φ gradually falls as s increases and is defined as the sliding coefficient of road adhesion when $|s| = 100\%$.

On a dry surface, $\varphi_{max} = 1.2\ \varphi_s$.

On a wet surface, $\varphi_{max} = 1.3\ \varphi_s$.

The coefficient of road adhesion depends mainly on the road texture and suface, the tire structure, the tread pattern, the material, the inflation pressure, the normal loading on the wheel, the travel speed of the vehicle, and so forth. Experimental data show that the adhesion coefficient of a radial tire is higher than that of a bias tire, and for a tire with a high inflation pressure is lower than that of a tire with a low inflation pressure. In addition, a low travel speed can produce a relatively high adhesion coefficient of the tire.

Problem 3.7

What are the tire cornering characteristics? Analyze the main factors that affect them, and draw a curve to show the relationship between the self-aligning torque and the slip angle.

Solution to Problem 3.7

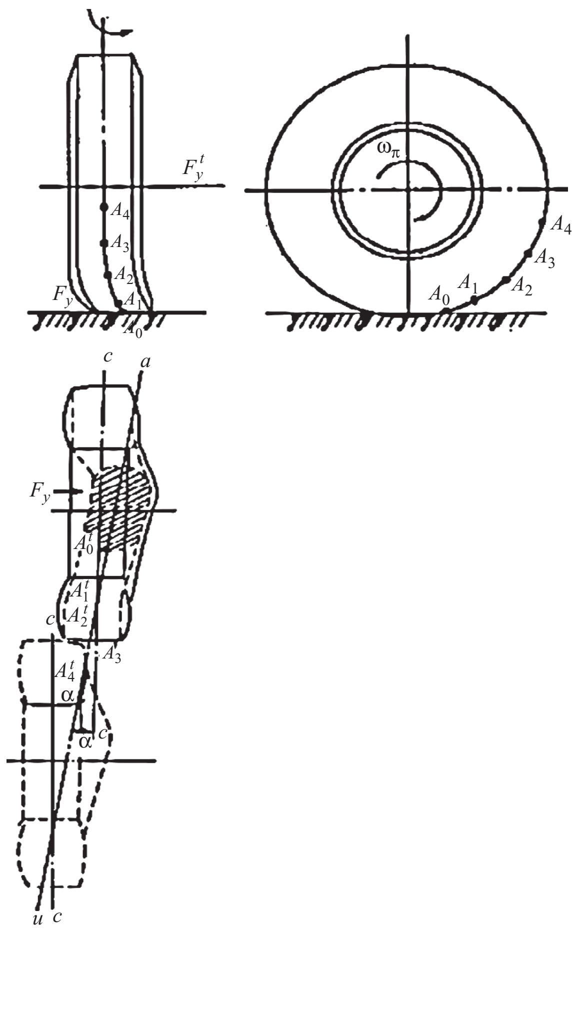


Figure 3.7 (a) Tire cornering characteristics.

Because of lateral wind or centrifugal force when driving, a side force of F_Y will be applied to the center of the tire. Correspondingly, a lateral reaction force also will be developed at the contact patch. This is called the cornering force F_Y. Because of the lateral elasticity of the tire, the tire begins to slip, and the centerline that is connected by the points A_0, A_1, A_2, A_3, A_4 on the wheel tread will be distorted. The angle between the path lines will be formed; this is the slip angle. This phenomenon is called the tire cornering characteristics.

The principal factors affecting tire cornering characteristics are the structure of the tire, the working condition, and the condition of the road.

1. Tire structure: A tire with a large size, a low profile, and a small flat rate has a high slip stiffness.
2. Work condition: Tire slip stiffness generally increases with an increase in normal force. It also increases with an increase in inflation pressure, and it does not change when the tire pressure reaches a certain high level.

Figure 3.7(b) shows a plot of experimental data.

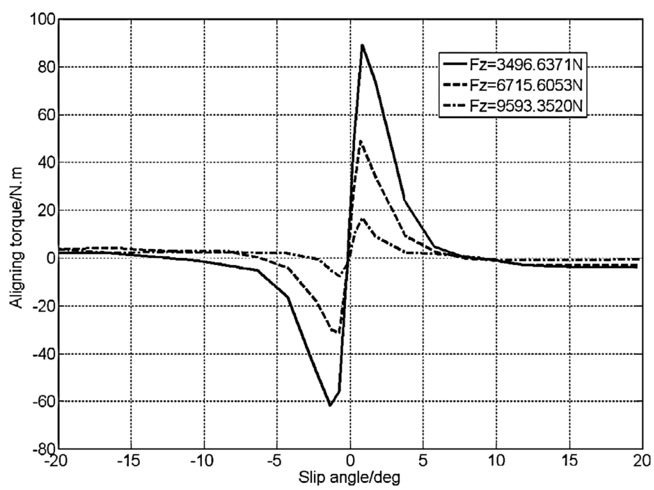


Figure 3.7 (b) Relationship between the aligning torque and the slip angle.

Experimental data show that the following is true:

The aligning torque first increases with an increase in slip angle. It reaches a maximum at the slip angle of 4° ~ 6° and then decreases with a further increase. It reaches zero at the slip angle of 10° ~ 16° and decreases to a negative value with a further increase in the slip angle. Furthermore, we can see that the aligning torque increases with an increase in the normal load.

Problem 3.8

The distribution function of the tire load is as follows:

$$\eta(u) = \begin{cases} \frac{3}{2}u(2-u) & 0 < u < 2 \\ 0 & u \le 0 \quad \text{or} \quad u \ge 2 \end{cases}$$

Calculate the tire cornering characteristic under a parabola distribution load of the tire ($\frac{F_y}{\varphi \cdot F_z} \sim \phi_y$ and $\frac{T_z}{\varphi \cdot F_z \cdot a} \sim \phi_y$).

Solution to Problem 3.8

The distribution of the normal force is

$$\eta(u) = \begin{cases} \frac{3}{2}u(2-u), & 0 < u < 2 \\ 0, & u \le 0 \quad \text{or} \quad u \ge 2 \end{cases} \tag{3.8.1}$$

This function complies with the two constraints of

$$\begin{cases} \int_0^2 u(u)\,du = 2 \\ \int_0^2 aq(u)\,du = Pa \end{cases} \tag{3.8.2}$$

$$\int_0^2 u\eta(u)\,du = 2\left(1 - \frac{\Delta}{a}\right) = 2(1-\theta) \qquad \theta = \frac{\Delta}{a} \tag{3.8.3}$$

Substitute Eq. 3.8.1 into Eq. 3.8.3 to get

$$\begin{cases} \int_0^2 u\eta(u)\,du = 2 \\ \theta = 0 \end{cases} \tag{3.8.4}$$

With the large cornering force and the slip angle, experimental data show that a partial sideslip has occurred on the tire because $q_y \ge \varphi \cdot q_z$ at the rearward patch. Also, φ is the friction coefficient, and the origin position can be defined by $q_y = \varphi \cdot q_z$.

Thus, one gets

$$k_y \cdot x \cdot tg\alpha = \varphi \cdot \frac{F_z}{2a}\eta(\mu) \tag{3.8.5}$$

Define the skid ratio as

$$\phi_y = \frac{K_y \cdot tg\alpha}{\varphi \cdot F_z} \tag{3.8.6}$$

Then,

$$\phi_y = \frac{\eta(\mu')}{\mu'} \tag{3.8.7}$$

Substitute Eq. 3.8.1 into Eq. 3.8.8 to get

$$\phi_y = 3/2(2 - u^*) \qquad (0 \le u^* \le 2) \tag{3.8.8}$$

or

$$u = \begin{cases} 2 - \frac{2}{3}\phi_y & (\phi_y \le 3) \\ 0 & (\phi_y \ge 3) \end{cases} \tag{3.8.9}$$

to get a nondimensional lateral force expression.

Substituting Eq. 3.8.1, the zero-order and first-order moment of $\eta(\mu)$, at the range of $0 \sim \mu$, as

$$m_0(\mu) = \int_0^{\mu} \eta(\mu)\,d\mu \tag{3.8.10}$$

$$m_1(\mu) = \int_0^{\mu} \mu \cdot \eta(\mu)\,d\mu \tag{3.8.11}$$

we get the equation

$$m_0(u^*) = 3/2^* \int_0^{u^*} u(2-u)\,du = 3/2^*\, u^{*2} - 1/2^*\, u^{*3}$$

Substituting the results of Eq. 3.8.10 and 3.8.11 gives

$$\frac{F_y}{\varphi \cdot F_z} = \begin{cases} \dfrac{\phi_y}{4} \times u^{*2} + 1 - \dfrac{m_0(u^*)}{2} & 0 \le u^* < 2 \\ \phi_y & u^* = 2 \end{cases} \tag{3.8.12}$$

The nondimensional lateral force expression is a single variable function.

We can get

$$\frac{F_y}{\varphi \cdot F_z} = \begin{cases} 1 - (1 - \phi_y/3)^3 & \phi_y \le 3 \\ 1 & \phi_y \ge 3 \end{cases} \tag{3.8.13}$$

Then use software (MATLAB) to draw a curve showing the relationship between $\dfrac{F_y}{\varphi \cdot F_z}$ and ϕ_y.

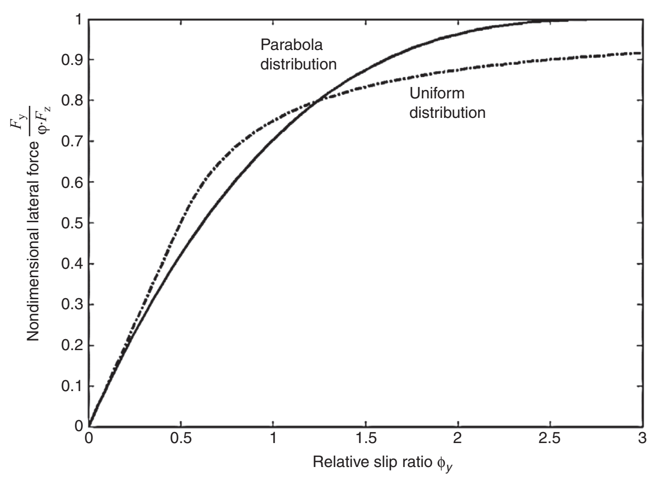


Figure 3.8 (a) Relationship between the nondimensional lateral force and the relative slip ratio.

From Figure 3.8(a), we can see that when $\phi_y \to 0$ or $\phi_y \to \infty$, the parabola distribution and the uniform distribution have the same result. And the two curves intersect at approximately $\phi_y = 1.24$, which shows when $\phi_y = 0 - 1.24$, the value of $\frac{F_y}{\varphi \cdot F_z}$ in the parabola distribution condition is smaller than in the uniform distribution condition. This result totally complies with the result of the Falia model.

Using the same method, we get the expression of the nondimensional total aligning torque.

$$\frac{T_z}{\varphi \cdot F_z \cdot a} = \begin{cases} 1/3 * \phi_y \left(1 - \phi_y/3\right)^3 & \phi_y \le 3 \\ 0 & \phi_y \ge 3 \end{cases} \tag{3.8.14}$$

Also, we get the curves as shown in Figure 3.8(b).

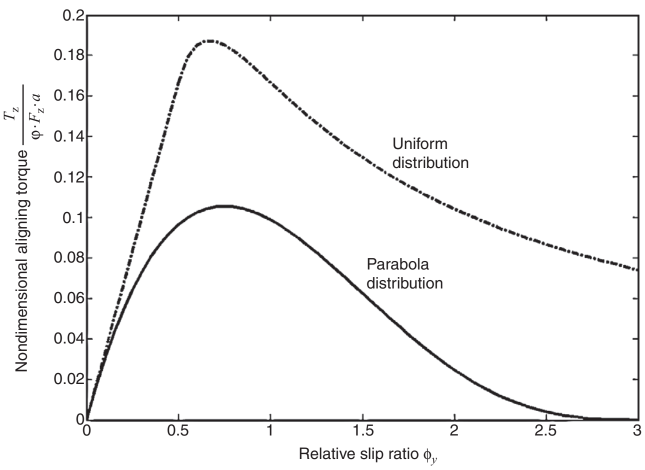


Figure 3.8 (b) The relationship between the nondimensional aligning torque and the relative slip ratio.

Problem 3.9

Establish a model, and then calculate the longitudinal force. For example, use the brush model as described below.

Solution to Problem 3.9

The rush model is a simplified theoretical model. This model considers the tire as a rigid base connected with a lot of bristles; through deformation of these bristles, the tire assumes the longitudinal force and the lateral force. The brush model states that the contact patch between the tire and the road is divided into an adhesion region, where the rubber is gripping the road, and a sliding region, where the rubber slides on the road surface. The total force generated by the tire then has components from these two regions.

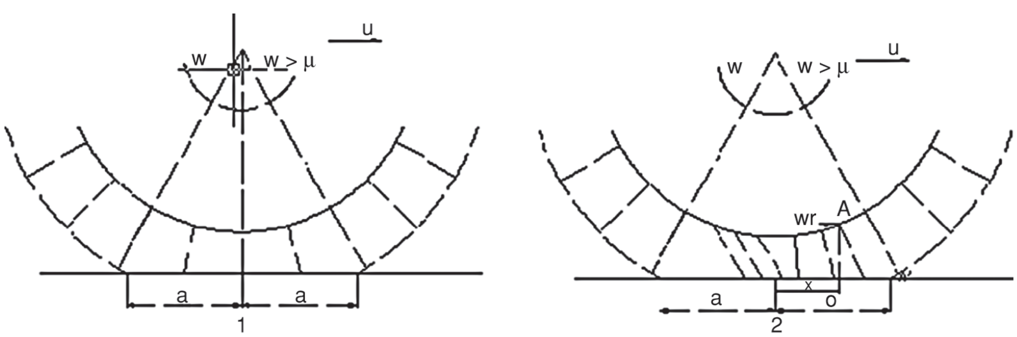


Figure 3.9 (a) Brush model.

In a state of pure rolling, as is shown on the left side of Figure 3.9(a), the tire contact patch length is 2a in the adhesion region. Because of the friction function of the ground, when wr > u, the bristle element at the end of the tire will be deformed, which causes the speed difference between the two ends, as is shown on the right side of Figure 3.9(a). It is assumed that the radius of the wheel is much larger than the length of the contact patch (r >> a). Therefore, the longitudinal deformation of the bristle element can be expressed as

$$\xi = (\text{wr-u})^*\Delta t \tag{3.9.1}$$

or

$$\xi = (\text{wr-u})^*\frac{\Delta t}{\Delta x}\Delta x = \frac{(\text{wr}-\text{u})}{\text{wr}}\Delta x \tag{3.9.2}$$

Define $\sigma_x = (\text{wr} - \text{u})/\text{wr}$ as the slip ratio, which is also called the amended slip ratio. The deformation of the bristle element can be expressed as

$$\xi = \sigma_x \Delta x$$

Assume that the longitudinal force is proportional to the deformation of the bristle element. Therefore, the force of the flexible unit can be expressed as

$$F_{ex} = \text{cex}\xi = c_{ex}\sigma_x(a - x) \tag{3.9.3}$$

where c_{ex} is the stiffness of the bristle element.

In the whole adhesion region, the longitudinal force can be found by integrating

$$F_x = \int_{-a}^{a} c_{ex}\xi \, dx = 2c_{ex}a^2\sigma_x \tag{3.9.4}$$

When the slip ratio is very small, we get the equations wr $\approx$ u and $\sigma \approx s$.

Therefore, the longitudinal force can be expressed as

$$d = |a| - |x_A| = c_{ex}\sigma_x/(\mu\lambda) \tag{3.9.5}$$

Equation 3.9.3 shows that the longitudinal force is proportional to the slip ratio of the wheel. However, this model is not concerned with the limit condition, which occurs when the longitudinal force has reached or exceeded the limits of the ground adhesion. Therefore, the model must be amended further.

The distribution of the normal force can be expressed as

$$F_{ez}(x) = \lambda(a^2 - x^2) \tag{3.9.6}$$

Through the equation

$$F_z = \int_{-a}^{a} \lambda(a^2 - x^2)dx$$

we get the value of λ.

When

$$F_{ex} \leq \mu F_{ez}(x)$$

the critical point A has divided the adhesion region into two parts: one is the adhesion region, and the other is the slipping region.

$$d = |a| - |x_A| = c_{ex}\sigma_x/(\mu\lambda) \tag{3.9.7}$$

Therefore, the whole longitudinal force can be expressed as

$$\begin{aligned} F_x &= \mu\lambda\int_{-a}^{x_A}(a^2 - x^2)dx + \int_{x_A}^{a} c_{ex}\sigma_{xc}(x-a)dx \\ &= 1/3\mu\lambda d^2(3a-d) + 1/2\mu_{st}\lambda d(2a-d)^2 \end{aligned} \tag{3.9.8}$$

From the figure, we know that the premise condition is $d \geq 2a$ if the wheel is in the pure slipping state. From this information, we can determine the critical slip ratio as

$$\sigma_{x,c} = 2a\mu\lambda/c_{ex}$$

Generally, because $\mu_{st} > \mu_{sd}$, we can change the former equation to

$$F_x = 1/3\mu_{sd}\lambda d^2(3a-d) + 1/2\mu_{st}\lambda d(2a-d)^2 \tag{3.9.9}$$

Now, the critical slip ratio is

$$\sigma_{x,c} = 2a\mu\lambda/c_{ex} \tag{3.9.10}$$

Finally, use software (MATLAB) to calculate the equation to draw a curve showing the relationship between the longitudinal force and the slip ratio. Use the following data:

$F_z = 4000$ N

$\mu_{st} = 1.0$

$\mu_{sd} = 0.6$

$a = 4$ cm

$\sigma_{xc} = 0.259$

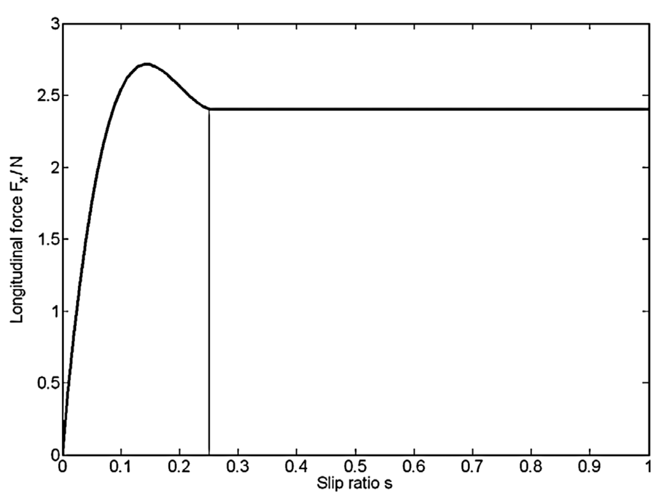


Figure 3.9 (b) Relationship between the lateral force and the slip ratio.

With use of the MATLAB program,

```
Fz = 4000, Us = 1.0, Ud = 0.6, a = 0.004;
k = ones (1,1/0.01+1);
e = 4.6875e + 007;
s = 0:0.01:1;
Ce = 1101100;
C = 8.6873e+005;
d = C.*s/(Ud*4.6875e+007);
m = length(d);
for i = 1:1:m;
   if d(I) > 2*a
      d(i) = 2*a;
end;
end;
Fx = 1/3*Ud*e.*d.^2.*(3*a.*k-d)+1/2*Us*e.*d.*(2*a.*k-d).^2;
plot(s,Fx)
```

Problem 3.10

Analyze the influence of rolling resistance to the axle load.

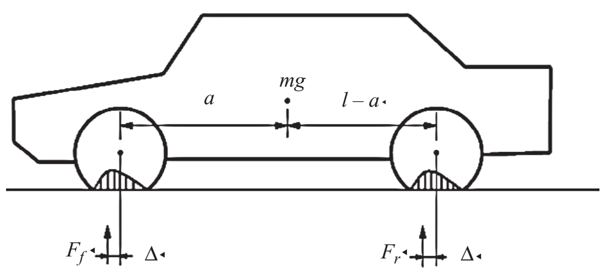


Figure 3.10 The influence of rolling resistance to the axle load.

Solution to Problem 3.10

If the wheelbase is ℓ, and the distance from the front axle to the centroid is a, then the distance from the front axle to the centroid is ℓ – a.

Thus, when the vehicle is static, we can calculate the load on each axle as follows:

Front axle:

$$F_{fs} = \frac{l - a}{l} mg \tag{3.10.1}$$

Rear axle:

$$F_{rs} = \frac{a}{l} mg \tag{3.10.2}$$

When the vehicle is running and the wheels are rolling, the deformation of the tires will cause these axle loads to change. As is shown in Figure 3.10, we get the following new results:

Front axle:

$$F_{fd} = \frac{l - a - \Delta}{l} mg = F_{fs} - \frac{\Delta}{l} mg \tag{3.10.3}$$

Rear axle:

$$F_{rd} = \frac{a + \Delta}{l} mg = F_{rs} + \frac{\Delta}{l} mg \tag{3.10.4}$$

where $\Delta = fR$. The distance between the normal force F and the wheel centerline f is the rolling resistance coefficient, and R is the rolling radius of the tires.

By substituting $\Delta = fR$ into the preceding equations, we get

$$F_{fd} = F_{fs} - \frac{fR}{l} mg \tag{3.10.5}$$

$$F_{rd} = F_{rs} + \frac{fR}{l} mg \tag{3.10.6}$$

From these results, we know that because of the rolling resistance, axle loads on both the front axle and the rear axle have changed. The loads on the front axle have decreased, and the loads on the rear axle have increased. This means that some of the loads are transferred from the front axle to the rear axle. The magnitude of the transferred loads is approximately $\frac{fR}{l}$ mg, and the bigger the rolling resistance coefficient is, the smaller the front axle load will become. Likewise, the rear axle load will become bigger at the same magnitude.

Problem 3.11

The load distribution function can be represented as

$$\eta(u) = \frac{n+1}{n}\left[1 - (u-1)^n\right]$$

Here, different n represent the different types of tires. However, when the load on the tire is not heavy, we can use another equation

$$\eta(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

It is much easier to calculate this second equation, as long as the coefficients of c_i are known and the load distribution function $\eta(u)$ is also known. Therefore, determine the coefficients of c_i if the tires are rolling on a road that is in good condition and the following parameters are given: the rolling resistance coefficient is f, the whole patch length is 2a, and the rolling radius is R.

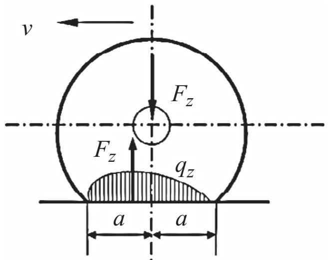


Figure 3.11 The load distribution of the rolling tire.

Solution to Problem 3.11

Because the tires are rolling on a road that is in good condition, we can assume that the rolling resistance coefficient is caused only by deformation of the tires. Therefore,

$$f = \frac{\Delta}{R}, \text{ that is, } \Delta = fR$$

From Eq. 3-40, we get

$$\begin{cases} \int_0^2 \left(c_0 + c_1 u + c_2 u^2 + c_3 u^3\right) du = 2 \\ \int_0^2 u\left(c_0 + c_1 u + c_2 u^2 + c_3 u^3\right) d\mu = 2\left(1 - \frac{\Delta}{a}\right) = 2\left(1 - \frac{fR}{a}\right) \end{cases} \tag{3.11.1}$$

Also, when u = 0, then $\eta(u) = 0$. When u = 2, then, $\eta(u) = 0$. Therefore,

$$\begin{cases} c_0 = 0 \\ c_0 + c_1 \cdot 2 + c_2 \cdot 2^2 + c_3 \cdot 2^3 = 0 \end{cases} \tag{3.11.2}$$

Solving these four equations will give the coefficients of c_i.

$$\begin{cases} c_0 = 0 \\ c_1 = 3 + 15\frac{fR}{a} \\ c_2 = -\frac{3}{2} - \frac{45}{2}\frac{fR}{a} \\ c_3 = \frac{15}{2}\frac{fR}{a} \end{cases} \tag{3.11.3}$$

Therefore, the load distribution function is

$$\eta(u) = \left(3 + 15\frac{fR}{a}\right)u + \left(-\frac{3}{2} - \frac{45}{2}\frac{fR}{a}\right)u^2 + \left(\frac{15}{2}\frac{fR}{a}\right)u^3 \tag{3.11.4}$$

Problem 3.12

When the vehicle is running on a wet surface, a hydroplaning phenomenon can easily occur when the speed of the vehicle reaches certain values. Such a condition is always extremely dangerous. Therefore, analyze the variation in the tire side force and the aligning torque when the hydroplaning phenomenon occurs. (Assume the slip angle α is constant.)

Solution to Problem 3.12

The distribution of the vertical load can be expressed as

$$q_z = \frac{F_z}{2a}\eta(u) \tag{3.12.1}$$

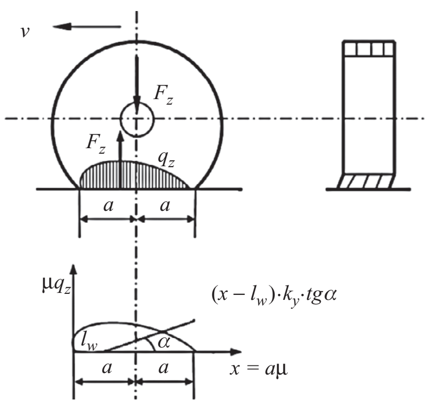


Figure 3.12 (a) The tire cornering characteristic under the hydroplaning condition.

If the length of the waterskiing area is l_w and the adhesion coefficient in the area of hydroplaning is assumed to be zero, then the lateral load distribution at the frontward patch is

$$\begin{cases} q_y = k_y \cdot (x - l_w) \cdot tg\alpha & l_w \le x \le u'a \\ q_y = 0 & x \le l_w \end{cases} \tag{3.12.2}$$

Partial sideslip has occurred on the tire because $q_y \ge \varphi\, q_z$ at the rearward patch. The origin position can be defined by $q_y = \varphi\, q_z$.

That is,

$$k_y(x - l_w)tg\alpha = \varphi\frac{F_z}{2a}\eta(u) \tag{3.12.3}$$

If u' is the answer to the above function, then the total side force can be expressed as

$$F_y = \int_{\frac{l_w}{a}}^{\mu'} a^2 k_y\left(u - \frac{l_w}{a}\right)tg\alpha \cdot u \cdot du + \int_{\mu'}^{2} a \cdot \varphi \cdot \frac{F_z}{2a} \cdot \eta(u) \cdot du \tag{3.12.4}$$

The total aligning torque is

$$M_z = \int_{\frac{l_w}{a}}^{\mu'} a^3 k_y\left(u - \frac{l_w}{a}\right)tg\alpha u^2 \cdot du + \int_{\mu'}^{2} a^2 u\varphi\frac{F_z}{2a}\eta(\mu)d\mu - F_y a \tag{3.12.5}$$

If all the parameters are given, it is easy to calculate the side force and the total aligning torque. Figure 3.12(b) shows the results of calculating by computer.

The following parameters are used in the calculations:

The vertical load is $F_z = 4000$ N.

The rolling radius of the tire is R = 0.3 m.

The whole patch length is 2a = 0.16 m.

The tire deformation rolling resistance coefficient is f = 0.015.

The lateral distribution stiffness of the rubber layer is $k_y = 3.5 \times 10^6$ N/m^2.

The slip angle is $\alpha = 4°$.

The friction coefficient is $\varphi = 0.6$.

The distribution function is $\eta(u) = c_0 + c_1u + c_2u^2 + c_3u^3$.

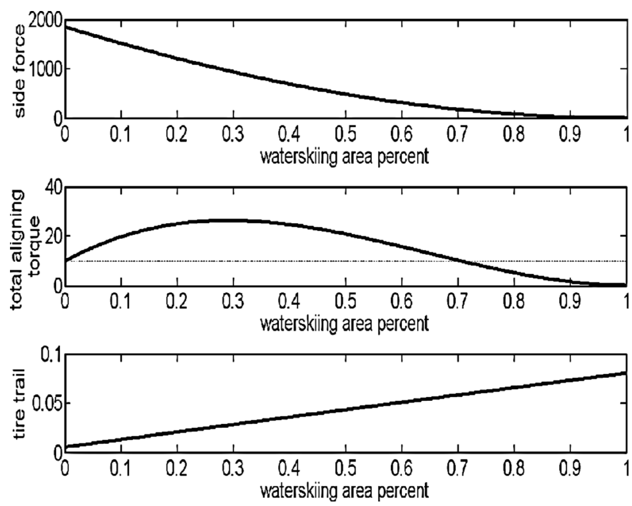


Figure 3.12 (b) The result of calculations.

From Figure 3.12(b), we know that the side force decreases when the area of hydroplaning becomes longer. When the area of hydroplaning covers the whole patch length, the tire will lose side force. However, the variation in the total aligning torque is not so simple. From Figure 3.12(b), we can determine the following information: The aligning torque does not decrease but increases when the area of hydroplaning becomes longer under a certain value. This occurs until the percent of hydroplaning reaches 30%, then the aligning torque stops increasing and begins to decrease. However, the aligning torque is still larger than when the non-hydroplaning occurs until the percent of the area of hydroplaning increases to 70%.

This occurs because the tire trail increases rapidly when the area of hydroplaning expands.

It is extremely dangerous when hydroplaning occurs, not only because of the decreased side force but also because of the increased aligning torque. Drivers always judge the situation of the tires through the aligning torque; thus, when the aligning torque increases, they often believe the road condition is very good, when, in fact, they are in danger.

Problem 3.13

When the cornering characteristic with lateral bending deformation of the tire case is considered, the shape function of the tire case deformation is needed. The shape function of the tire case deformation can be represented as

$$\xi(u) = 2u(2 - X_e u)/(4 - 8/3X_e)$$

where X_e is a constant determined by the type of tire. Analyze the influence of the tire case bending stiffness on the tire side force and aligning torque. (Assume the slip angle α is small.)

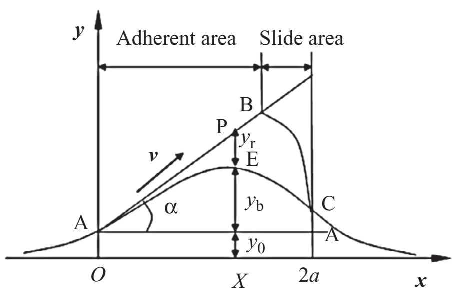


Figure 3.13 (a) Cornering characteristics with lateral bending deformation of the tire case.

Solution to Problem 3.13

With a small slip angle α, there is no side skidding on the whole patch. Thus, the total cornering force can be expressed as

$$\begin{aligned} F_y &= \int_0^{2a} q_r \cdot dx \\ &= \int_0^{2} a^2 u \cdot k_{yr} \cdot tg\alpha \cdot du - \int_0^{2} a^2 u \cdot k_{yr} \cdot \frac{F_y}{k_{yb} \cdot au} \xi(u) \cdot du \end{aligned} \tag{3.13.1}$$

After integration,

$$F_y = 2a^2 \cdot k_{yr} \cdot tg\alpha - a\frac{k_{yr}}{k_{yb}} F_y \int_0^2 \xi(u)du \tag{3.13.2}$$

Solving this function gives

$$F_y = \frac{2a^2 \cdot k_{yr} \cdot tg\alpha}{1 + a\dfrac{k_{yr}}{k_{yb}} \displaystyle\int_0^2 \xi(u)du} \tag{3.13.3}$$

Note that $\xi(u)$ can be represented as

$$\xi(u) = 2u(2 - X_e u)/(4 - 8/3X_e) \tag{3.13.4}$$

Then,

$$\int_0^2 \xi(u)du = 2 \quad \text{and} \quad \int_0^2 u\xi(u)du = \frac{6X_e - 8}{2X_e - 3}$$

Substitute into the above equation the following:

$$\int_0^2 \xi(u)\,du = 2$$

$$K_{yr} = 2a^2 k_{yr}$$

$$K_{yb} = a\delta_{yb}$$

Then,

$$F_y = \frac{K_{yr}}{1 + \dfrac{K_{yr}}{K_{yb}}} tg\alpha = \frac{K_{yb}K_{yr}}{K_{yb} + K_{yr}} tg\alpha \tag{3.13.5}$$

and the aligning torque can be expressed as

$$\begin{aligned} T_z &= \int_0^2 a^2 \cdot q_y \cdot u\,du - F_y \cdot a \\ &= \int_0^2 a^3 \cdot u^2 \cdot k_{yr}\left[tg\alpha - F_y/(a \cdot k_{yb}) \cdot \xi(u)/u\right]du - F_y \cdot a \\ &= \frac{4}{3}aK_{yr}tg\alpha - \frac{a}{2}\frac{K_{yr}}{K_{yb}}F_y\int_0^2 u\xi(u)\,du - a\frac{K_{yr}K_{yb}}{K_{yr} + K_{yb}}tg\alpha \qquad (3.13.6) \\ &= \frac{4}{3}aK_{yr}tg\alpha - \frac{a}{2}\frac{K_{yr}^2}{K_{yb} + K_{yr}} \cdot \frac{6X_e - 8}{2X_e - 3}tg\alpha - a\frac{K_{yr}K_{yb}}{K_{yr} + K_{yb}}tg\alpha \\ &= \left(\frac{4}{3}K_{yr} - \frac{1}{2}\frac{K_{yr}^2}{K_{yb} + K_{yr}} \cdot \frac{6X_e - 8}{2X_e - 3} - \frac{K_{yr}K_{yb}}{K_{yr} + K_{yb}}\right)a \cdot tg\alpha \end{aligned}$$

If all necessary parameters are given, it is easy to calculate the side force and the total aligning torque. Figure 3.13(b) shows the results of calculating by computer.

The parameter used in the calculation is the following:

The whole patch length is $2a = 0.16$ m.

The slip stiffness of the tire tread is $K_{yr} = 45{,}000$.

The bend stiffness of the tire case K_{yb} changes from 20,000 to ~40,000.

The slip angle is $\alpha = 4°$.

The shape function coefficient is $X_e = 0.8$.

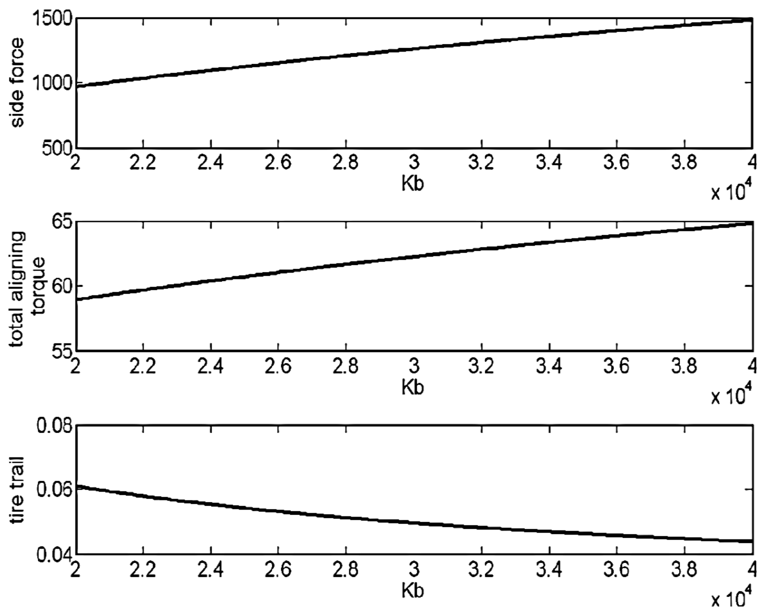


Figure 3.13 (b) Results of the calculations.

From Figure 3.13(b), we know that the side force increases when the bending stiffness of the tire case K_{yb} increases, and the aligning torque has the same trend. However, the tire trail has the opposite trend. When the bending stiffness of the tire case K_{yb} increases, the tire trail decreases. This means that when the side force is at the same magnitude, the tire with the softer tire case will have the larger aligning torque. Furthermore, this is the reason why a radial ply tire always gets more aligning torque than a bias ply tire.

Problem 3.14

Based on the parameters given in Problem 3.13, write a piece of the MATLAB software program that can generate a two-dimensional curve representing the relationship between the steering angle and the rolling resistance of the tire. (The steering angle ranges from 0 to 30°.)

Solution to Problem 3.14

MATLAB program:

```
m = 1820, la = 1.463, lb = 1.585, f0 = 0.0165, g = 9.8, v0 = 36;

afdeg = 4;

ardeg = 4;

xdeg = 0:0.001:30;

af = afdeg*pi/180;

ar = ardeg*pi/180;

x = xdeg*pi/180;

v = v0/3.6;

r0 = (la+lb).*sin(pi/2-x+af)./sin(x-af+ar);

r = sqrt(lb.^2+r0.^2-2*lb.*r0.*cos(pi/2-ar));
```

```
b = asin(lb.*cos(ar)./r);
y = pi/2-b+ar;
fyf = m.*v.^2.*sin(b-ar)./(r.*sin(x));
fyr = m.*v.^2.*(cos(b-ar)-sin(b-ar)./tan(x))./r;
dff = sin(af).*fyf;
dfr = sin(ar).*fyr;
df = (dff+dfr)./(m.*g);
fr = f0+df;
f = m.*g.*fr;
plot(xdeg,f);
```

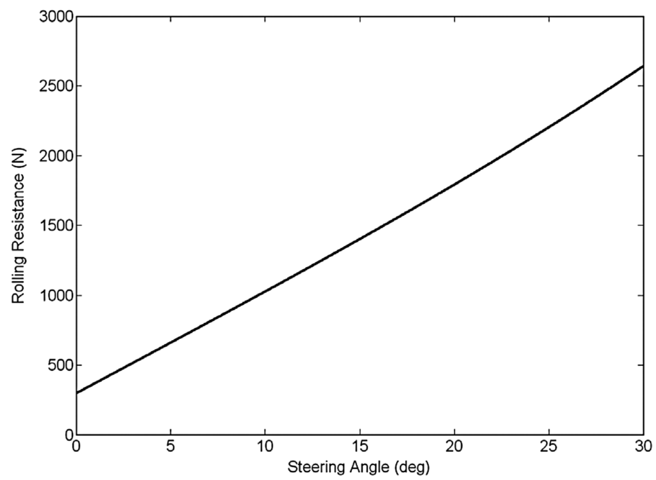


Figure 3.14 Rolling resistance changes with the steering angle.

Problem 3.15

The radius of a vehicle tire is R = 0.3 m, the camber angle is $\gamma = 1°$, the camber stiffness of the tire carcass is $K_c = 730{,}800$ N/m, the length of the contact patch is 2a = 0.06 m, and the lateral distributed stiffness per length of tire tread is $k_y = 4 \times 10^8$ N/m. Calculate the lateral force with camber.

Solution to Problem 3.15

The broken line shown in Figure 3.15 represents the deformed part of the tire. A coordinate system Oxz is located at the left side of the contact patch with a length of 2a.

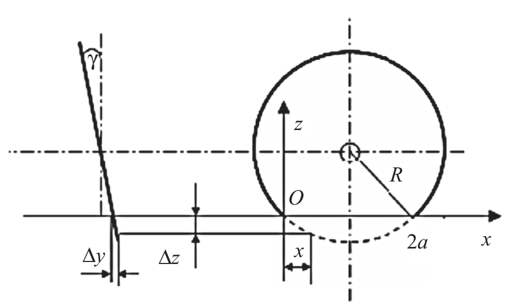


Figure 3.15 Lateral force model.

Therefore, the vertical deformation of the tire is

$$\begin{aligned}\Delta z &= \Delta z_m \mu(2-\mu) \\ &= \frac{a^2}{2R}\mu(2-\mu) \\ &= 0.0015\mu(2-\mu)\end{aligned} \tag{3.15.1}$$

Corresponding to x, the lateral deformation is

$$\Delta y = \Delta z \cdot \text{tg}\gamma = 2.618 \times 10^{-5}\mu(2-\mu) \tag{3.15.2}$$

The lateral stress at the site x is

$$q_y = k_y \cdot \Delta y = 10472\mu \cdot (2-\mu) \tag{3.15.3}$$

The lateral force with camber is

$$\begin{aligned}F_{yr0} &= \int_0^{2a} q_y dx \\ &= 10{,}472\int_0^{2a}\mu(2-\mu)dx \\ &= 10{,}472 \times \frac{4}{3} \times 0.03 \\ &= 419 \text{ N}\end{aligned} \tag{3.15.4}$$

The cornering stiffness of the tire is

$$\begin{aligned}K_y &= 2a^2 k_y \\ &= 2 \times 0.03^2 \times 4 \times 10^8 \\ &= 720{,}000 \text{ N/m}\end{aligned} \tag{3.15.5}$$

The camber stiffness is

$$\begin{aligned}K_r &= \frac{a}{3R}K_y \\ &= \frac{0.03 \times 720{,}000}{3 \times 0.3} \\ &= 24{,}000 \text{ N/m}\end{aligned} \tag{3.15.6}$$

With consideration of the roll elastic deformation of the tire in the modeling of a cambered tire, the roll force of the tire due to the roll of tire should be

$$\begin{aligned}F_{yr} &= \frac{K_r \cdot K_c}{K_r + K_c} \cdot \text{tg}\gamma \\ &= \frac{24{,}000 \times 730{,}800}{24{,}000 + 730{,}800} \times \text{tg}1° \\ &= 406 \text{ N}\end{aligned} \tag{3.15.7}$$

Problem 3.16

A vehicle is in a drive state. The circumference speed of the wheel is $v_c = 70$ km/h, the longitudinal stiffness of the tire is $K_s = 1{,}001{,}100$ N/m, the lateral stiffness is $K_\alpha = 975{,}000$ N/m, the longitudinal velocity of the vehicle is $v_x = 65$ km/h, the lateral velocity is $v_y = 4$ km/h, the static friction coefficient between the tire and the road surface is $\varphi_0 = 0.9$, the friction coefficient is $\varphi_1 = 0.6$ when the slip ratio is $S_1 = 8\%$, the vertical load is $F_z = 4000$ N, and the length of the contact patch of the tire is $l = 0.06$ m. Calculate the longitudinal force, the lateral force, and the self-aligning moment of the driving tire by taking advantage of the C.G. Gim theoretical model.

Solution to Problem 3.16

The slip angle of the tire is

$$\alpha = \arctan\frac{v_y}{v_x} = \arctan\frac{4}{65} = 3.52° \tag{3.16.1}$$

The longitudinal slip ratio is

$$S_s = \frac{v_c - v_x}{v_c} = \frac{70 - 65}{70} \times 100\% = 7.14\% \tag{3.16.2}$$

The lateral slip ratio is

$$\begin{aligned} S_\alpha &= (1 - S_s) \cdot |\tan\alpha| \\ &= (1 - 7.14\%) \times |\tan 3.52°| \\ &= 5.71\% \end{aligned} \tag{3.16.3}$$

The associated parameter of the slip ratio is

$$\begin{aligned} S_{s\alpha} &= \sqrt{S_s^2 + S_\alpha^2} \\ &= \sqrt{(7.14\%)^2 + (5.71\%)^2} \\ &= 9.15\% \end{aligned} \tag{3.16.4}$$

Then the comprehensive adhesion coefficient of the tire is found as

$$\begin{aligned} \varphi &= \varphi_0 + (\varphi_1 - \varphi_0) \times S_{sa} \\ &= 0.95 + (0.75 - 0.95) \times 9.15\% \\ &= 0.9317 \end{aligned} \tag{3.16.5}$$

Therefore, the longitudinal adhesion coefficient of the tire is

$$\begin{aligned} \varphi_x &= \varphi \cdot \frac{S_s}{S_{s\alpha}} \\ &= \varphi \cdot \cos\alpha \\ &= 0.9317 \times \cos 3.52° \\ &= 0.9299 \end{aligned} \tag{3.16.6}$$

The lateral adhesion coefficient of the tire is

$$\begin{aligned}\varphi_y &= \varphi \cdot \frac{S_\alpha}{S_{s\alpha}} \\ &= \varphi \cdot \sin\alpha \\ &= 0.9317 \times \sin 3.52° \\ &= 0.0572\end{aligned} \tag{3.16.7}$$

The critical point of rolling and slip in the contact patch is

$$\begin{aligned}S_n &= \frac{1}{3\varphi F_z}\sqrt{\left(K_s S_s\right)^2 + \left(K_\alpha S_\alpha\right)^2} \\ &= \frac{\sqrt{\left(79981 \times 7.14\%\right)^2 + \left(65836 \times 5.71\%\right)^2}}{3 \times 0.9317 \times 4000} \\ &= 0.6118\end{aligned} \tag{3.16.8}$$

The slip critical point is

$$S_{sc} = \frac{3\varphi F_z}{K_s} = \frac{3 \times 0.9317 \times 4000}{79981} = 0.1398 \tag{3.16.9}$$

The cornering critical point is

$$\begin{aligned}S_{ac} &= \frac{K_s}{K_\alpha}\sqrt{S_{sc}^2 - S_s^2} \\ &= \frac{79{,}981}{65{,}836}\sqrt{0.1398^2 - 0.0714^2} \\ &= 0.1460\end{aligned} \tag{3.16.10}$$

For $S_s < S_c$, $S_a < S_{ac}$,

Set $l_n = 1 - S_n = 1 - 0.6118 = 0.3882$

Thus, the longitudinal force between the tire and the road surface is

$$\begin{aligned}F_x &= K_s S_s l_n^2 + \varphi_x F_z\left(1 - 3l_n^2 + 2l_n^3\right) \\ &= 79{,}981 \times 7.14\% \times 0.3882^2 \\ &\quad + 0.9299 \times 4000 \times \left(1 - 3 \times 0.3882^2 + 2 \times 0.3882^3\right) \\ &= 3333.8\ \text{N}\end{aligned} \tag{3.16.11}$$

The lateral force between the tire and the road surface is

$$\begin{aligned}F_y &= K_a S_a l_n^2 + \varphi_y F_z\left(1 - 3l_n^2 + 2l_n^3\right) \\ &= 65836 \times 5.71\% \times 0.3882^2 + 0.0572 \\ &\quad \times 4000 \times \left(1 - 3 \times 0.3882^2 + 2 \times 0.3882^3\right) \\ &= 719.1\ \text{N}\end{aligned} \tag{3.16.12}$$

The self-aligning moment is

$$
\begin{aligned}
T_Z &= \left[K_a S_a\left(-\frac{1}{2}+\frac{2}{3}l_n\right)+\frac{3}{2}\varphi_y F_z S_n^2\right]\cdot 1\cdot l_n^2 \\
&= \left[65836\times 5.71\%\times\left(-\frac{1}{2}+\frac{2}{3}\times 0.3882\right)\right. \\
&\quad \left.+\frac{3}{2}\times 0.0572\times 4000\times 0.6118^2\right]\times 0.06\times 0.3882^2 \\
&= -7.04\mathrm{N}\cdot\mathrm{m}
\end{aligned}
\tag{3.16.13}
$$

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Chapter 4

Ride Dynamics

Problem 4.1

Describe the factors that influence ride quality.

Solution to Problem 4.1

Ride quality is about passengers' comfort while sitting in a vehicle under a vibration environment. Various vibration sources pass through the structure of the vehicle and transfer to the passengers (receivers), as shown in Figure P4.1. Thus, the factors that influence ride quality come from two sources: (1) vibration sources, and (2) transfer paths.

Figure P4.1 Sources of factors that influence ride quality.

These sources include outside and internal sources. Primary outside sources are road vibration and wind-induced vibration, but the contribution to ride quality from wind is insignificant compared with road input. Road excitation depends on an uneven profile of the road and the speed at which the vehicle is traveling. The internal sources come from the engine and other systems connected to the engine, such as the exhaust system and the driveline system. These internal sources depend on the speed of the engine.

The paths refer to the vehicle body structure, the suspension, and the seat. Road vibration is transferred to the vehicle body through the suspension of the vehicle. Internal vibration is transferred to the body directly or through the frame or subframe of the vehicle. Ultimately, all vibration is transferred to the passengers through the seats of the vehicle.

Problem 4.2

Explain why the firing pulsation is the most important engine excitation. List the firing order for four-, six-, eight-, and twelve-cylinder engines, respectively.

Solution to Problem 4.2

Engine excitations include the firing pulsation and the inertia force (moment).

The working process of a four-stroke engine includes four steps: intake, compression, combustion (firing), and exhaust. Two cycles are necessary to complete an entire working process. In one cycle, only half of the "firing" occurs. For a four-cylinder engine, four "half-firings" (i.e., two complete firings) occur. The corresponding firing order is second order. The firing order for four-, six-, eight-, and twelve-cylinder engines is as follows:

Engine Cylinder Number	Firing Order
4	Second
6	Third
8	Fourth
12	Sixth

Problem 4.3

Analyze the dominant orders caused by inertia force.

Solution to Problem 4.3

To analyze the dominant orders caused by inertia force and moment, we need to analyze the motion of the crank-rod mechanism. Figure P4.2 shows a simplified crank-rod structure. Translational motion of the piston is transformed to rotational motion of the crankshaft.

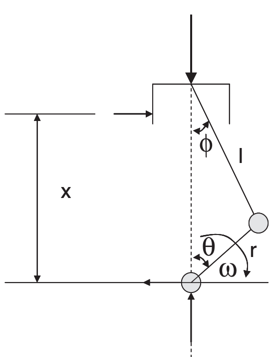

Figure P4.2 Crankshaft-rod structure.

The displacement of the piston can be expressed as

$$x = r\cos\theta + l\cos\phi \tag{4.3.1}$$

From Figure 4.2, we obtain

$$l\sin\phi = r\sin\theta \tag{4.3.2}$$

Assume

$$\lambda = \frac{r}{l} \tag{4.3.3}$$

Substitute Eqs. 4.4.2 and 4.4.3 into Eq. 4.4.1 to obtain

$$x = r\cos\theta + l\sqrt{1 - \lambda^2 \sin^2\theta} \tag{4.3.4}$$

Differentiate Eq. 4.4.4 to obtain the acceleration of the piston as

$$\begin{aligned} \ddot{x} = -\omega^2 r\Big\{\cos\theta + \lambda\left(1 - \lambda^2\sin^2\theta\right)^{-1/2}\cos 2\theta \\ + \frac{\lambda^3}{8}\left(1 - \lambda^2\sin^2\theta\right)^{-3/2}\left(1 - \cos(4\theta)\right)\Big\} \end{aligned} \tag{4.3.5}$$

where $\omega = d\theta/dt$ is the angular speed of the crankshaft.

Usually, $\lambda \le 0.35$. Therefore, $\frac{\lambda^3}{8} \approx 0$ and $\left(1 - \lambda^2\sin^2\theta\right) \approx 1$. Equation 4.4.5 can be simplified as

$$\ddot{x} \approx -\omega^2 r\{\cos\theta + \lambda\cos 2\theta\} \tag{4.3.6}$$

Thus, the inertia force of the piston is obtained as

$$\begin{aligned} F &= m\ddot{x} \\ &= m\omega^2 r\{\cos\theta + \lambda\cos 2\theta\} \\ &= m\omega^2 r\cos\theta + m\omega^2 r\lambda\cos 2\theta \end{aligned} \tag{4.3.7}$$

In Eq. 4.4.7, only two items are found. One is $F_1 = m\omega^2 r\cos\theta$, and the other is $F_2 = m\omega^2 r\lambda\cos 2\theta$. F_1 contains only $\cos\theta$; thus, it is the first-order force. F_2 contains only $\cos 2\theta$; thus, it is the second-order force.

Problem 4.4

For a four-cylinder engine, the powerplant weight is 220 kg, and the idle speed is 700 rpm. Assume that the powerplant is supported uniformly by three mounts. The rate of the mounts is the same. What is the practical stiffness range for the mounting system?

Solution to Problem 4.4

The firing order for a four-cylinder engine is second order. The firing frequency at an idle speed of 700 rpm is

$$f_{firing} = \frac{700}{60} * 2 = 23.3 \text{ Hz}$$

The firing frequency is the excitation frequency. According to Eq. 4.26 in the textbook, the powerplant roll mode frequency must be satisfied as

$$f_{roll} = \frac{f_{firing}}{2} \sim \frac{f_{firing}}{3}$$

Hence, f_{roll} should be between 7.77 and 11.65 Hz. Here, 11.65 Hz is the maximum roll mode frequency. Thus, the total maximum stiffness of the mounts should be

$$\begin{aligned} K &= M\omega^2 \\ &= M(2\pi f)^2 \\ &= 220 * (2 * 3.14 * 11.65)^2 \\ &\approx 1{,}179{,}000 \text{ N/m} \end{aligned}$$

The stiffness of each mount is

$$K/3 = 393{,}000 \text{ N/m}$$

To satisfy the isolation requirement, the mount should be designed softer than the calculated stiffness.

Problem 4.5

A vehicle has a six-cylinder engine, and its body response is measured. The response spectrum is dominated mainly by second order and third order when the vehicle travels at wide open throttle (WOT) condition. Explain the vibration sources to the body response.

Solution to Problem 4.5

The firing order of a six-cylinder engine is third order. Usually, the firing order and its harmonic orders are dominant for vehicle vibration. In this instance, the measured data showed that the second order and the third order were dominant. The third order definitely is from the engine firing force and moment. The second order has no relation to the engine firing forces.

As was introduced in Problem 4.3, the powerplant inertia force has components of two orders: first order and second order. By the same analysis method as is used for inertia force, the powerplant inertia moment contains only first-order and second-order components. Thus, the second-order excitation measured in this problem is from the powerplant inertia force and inertia moment.

Problem 4.6

The idle speed for a four-cylinder engine is 600 rpm. Determine the natural frequency of the powerplant roll mode.

Solution to Problem 4.6

The idle speed of 600 rpm is the lowest engine speed. The firing frequency for a four-cylinder engine running at 600 rpm is

$$f_{firing} = \frac{600}{60} * 2 = 20 \text{ Hz}$$

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According to design requirements, the ratio between the firing frequency and the roll mode frequency must satisfy the requirement of

$$f_{roll} = \frac{f_{firing}}{2} \sim \frac{f_{firing}}{3}$$

Thus, the corresponding roll mode frequency is

$$f_{roll} = \frac{f_{firing}}{2} \sim \frac{f_{firing}}{3} = \frac{20}{2} \sim \frac{20}{30} = 10 \sim 6.67 \text{ Hz}$$

The natural frequency of the powerplant roll mode should be between 6.67 and 10 Hz.

Problem 4.7

Plot the relation among engine speed, frequency, and order from first order to fourth order at increments of one-half order.

Solution to Problem 4.7

The relation among engine speed, firing frequency, and order is given by Eq. 4.21 in the textbook. Assume that the engine speed is 1000 to 6000 rpm. The relation of the three parameters can be listed as

rpm	1st Order	1.5th Order	2nd Order	2.5th Order	3rd Order	3.5th Order	4th Order
1000	16.7	25	33.3	41.7	50	58.3	66.7
2000	33.3	50	66.7	83.3	100	116.7	133.3
3000	50	75	100	125	150	175	200
4000	66.7	100	133.3	166.7	200	233.3	266.7
5000	83.3	125	166.67	208.3	250	291.7	333.3
6000	100	150	200	250	300	350	400

The relation is plotted in Figure P4.3.

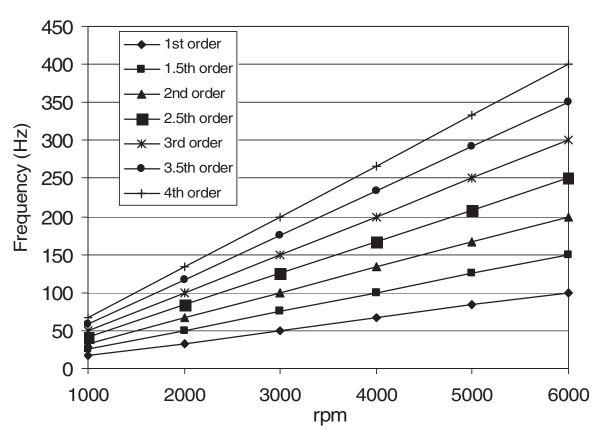

Figure P4.3 Plot of the relation among engine speed, firing frequency, and order.

Problem 4.8

For a V-8 engine, the maximum engine speed is 6500 rpm. Determine the minimal exhaust hanger bracket frequency.

Solution to Problem 4.8

The exhaust hanger bracket frequency is determined by Eq. 4.41 in the textbook. For the V-8 engine and maximum engine speed 650 rpm, the minimal hanger bracket frequency should be

$$f_{min} = \frac{6500 * 8}{120} = 433 \text{ Hz}$$

Problem 4.9

The driveshaft maximum imbalance force must be controlled to less than 50 N. The maximum shaft speed is 3500 rpm. Determine the static balance of the system.

Solution to Problem 4.9

The static balance is defined as the product of imbalance mass, m, and the distance from the imbalance mass to the shaft rotation center, r, expressed by $I_{static} = mr$.

The driveshaft imbalance force, $F_{imbalance}$, is expressed as

$$F_{imbalance} = I_{static}\omega^2$$

The rotation frequency that corresponds to the maximum shaft speed of 3500 rpm is

$$\omega = 2\pi f = 2 * 3.14 * \frac{3500}{60} = 366.5 \text{ rad/s}$$

The system static balance should be

$$\begin{aligned} I_{static} &= \frac{F_{imbalance}}{\omega^2} \\ &= \frac{50}{366.5^2} \\ &= 3.722 * 10^{-4} \text{ kg} - \text{m} \\ &= 37.2 \text{ g} - \text{cm} \end{aligned}$$

Problem 4.10

If only a disturbance force f_b is applied on the axle (unsprung mass) and other inputs are neglected, as is shown in Figure P4.4, what is the transmissibility between the body response and the disturbed force?

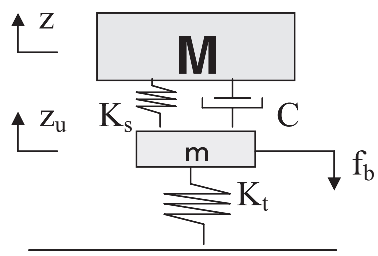

Figure P4.4 Disturbance force applied on the axle.

Solution to Problem 4.10

The dynamic equations for the system can be written as

$$M\ddot{z} + C\dot{z} + K_s z = K_s z_u + C\dot{z}_u \tag{4.10.8}$$

$$m\ddot{z}_u + C\dot{z}_u + \left(K_s + K_t\right) z_u = K_s z + C\dot{z} + f_b \tag{4.10.9}$$

The displacement solutions and the force excitation can be assumed as

$$\begin{aligned} z &= Ze^{j\omega t} \\ z_u &= Z_u e^{j\omega t} \\ f_b &= F_b e^{j\omega t} \end{aligned} \tag{4.10.10}$$

Substitute Eq. 4.10.10 into Eqs. 4.10.8 and 4.10.9 to obtain

$$\left(K_s - M\omega^2 + jC\omega\right) Z = \left(K_s + jC\omega\right) Z_u \tag{4.10.11}$$

$$\left(K_s + K_t - m\omega^2 + jC\omega\right) Z_u = \left(K_s + jC\omega\right) Z + F_b \tag{4.10.12}$$

Through substitution of Eq. 4.10.11 into Eq. 4.10.12, the transmissibility between the body response and the disturbed force can be determined as

$$\frac{Z}{F_b} = \frac{\left(K_s + jC\omega\right)}{\left(K_s + K_t - m\omega^2 + jC\omega\right)\left(K_s - M\omega^2 + jC\omega\right) - \left(K_s + j\omega C\right)^2} \tag{4.10.13}$$

Problem 4.11

Calculate the natural frequencies and the modal shapes of the quarter model shown in Figure 4.28 in the textbook with the following data:

Sprung mass:	M = 1800 kg
Unsprung mass:	m = 180 kg
Sprung stiffness:	Ks = 80,000 N/m
Tire stiffness:	Kt = 750,000 N/m

Solution to Problem 4.11

The sprung mass system natural frequency is

$$\omega_s \approx \sqrt{\frac{K_s}{M}} = \sqrt{\frac{80000}{1800}} = 6.67 \text{ rad/s}$$

and

$$f_s = \frac{\omega_s}{2\pi} = 1.06 \text{ Hz}$$

According to Eq. 4.55 in the textbook, the natural frequencies of the quarter model are expressed as

$$\omega_1 \approx 0.95\omega_s = 0.95 * 6.67 = 6.34 \text{ rad/s} \quad \text{and} \quad f_1 = \frac{\omega_1}{2\pi} = 1.01 \text{ Hz}$$

$$\omega_2 \approx 10\omega_s = 10 * 6.67 = 66.7 \text{ rad/s} \quad \text{and} \quad f_2 = \frac{\omega_2}{2\pi} = 10.62 \text{ Hz}$$

According to Eq. 4.56 in the textbook, the first and second modal shapes are obtained as

First modal shape:

$$\left(\frac{Z}{Z_u}\right)_1 = \frac{\omega_s^2}{\omega_s^2 - \omega_1^2} = \frac{6.67^2}{6.67^2 - 6.34^2} = 10.36$$

Second modal shape:

$$\left(\frac{Z}{Z_u}\right)_2 = \frac{\omega_s^2}{\omega_s^2 - \omega_2^2} = \frac{6.67^2}{6.67^2 - 66.7^2} = -0.01$$

The modal shapes are the same as Figure 4.29 in the textbook and are replotted in Figure P4.5.

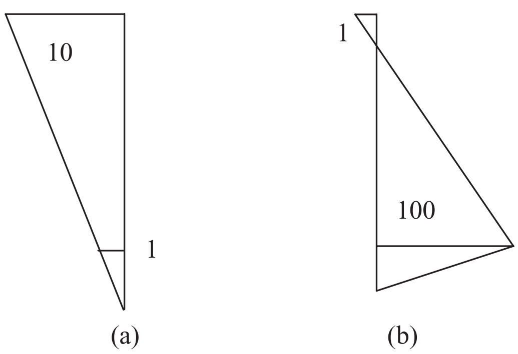

Figure P4.5 (a) First mode, and (b) second mode.

PROBLEM 4.12

Calculate the pure bounce and pitch frequencies and their oscillation centers for a vehicle with the following characteristics.

Front ride rate: $k_f = 32{,}000$ N/m

Rear ride rate: $k_r = 38{,}000$ N/m

Sprung mass: $M = 1400$ kg

Radius of gyration: $r_y = 1.1$ m

Distance between the front axle and the vehicle CG: $l_1 = 1.3$ m

Distance between the rear axle and the vehicle CG: $l_2 = 1.7$ m

Solution to Problem 4.12

According to Eqs. 4.76, 4.77, and 4.78 in the textbook, the parameters A1, A2, and A3 are calculated as

$$A_1 = \frac{k_f + k_r}{M} = \frac{32000 + 38000}{1400} = 50$$

$$A_2 = \frac{k_f l_1 - k_r l_2}{M} = \frac{32000 * 1.3 - 38000 * 1.7}{1400} = -16.43$$

$$A_3 = \frac{k_f l_1^2 + k_r l_2^2}{I_y} = \frac{k_f l_1^2 + k_r l_2^2}{M r_y^2} = \frac{32000 * 1.3^2 + 38000 * 1.7^2}{1400 * 1.1^2} = 96.75$$

The pure bounce frequency, f_{nz}, is

$$f_{nz} = \frac{\omega_{nz}}{2\pi} = \frac{\sqrt{A_1}}{2\pi} = \frac{\sqrt{50}}{2\pi} = 1.13 \text{ Hz}$$

The pure pitch frequency, $f_{n\theta}$, is

$$f_{n\theta} = \frac{\omega_{n\theta}}{2\pi} = \frac{\sqrt{A_3}}{2\pi} = \frac{\sqrt{96.75}}{2\pi} = 1.57 \text{ Hz}$$

Usually, the bounce and pitch modes are coupled to each other. The natural frequencies of the system can be obtained by Eqs. 4.88 and 4.89 in the textbook as

$$\begin{aligned}\omega_{n1}^2 &= \frac{A_1 + A_3}{2} - \sqrt{\frac{1}{4}\left(A_1 - A_3\right)^2 + \frac{A_2^2}{r_y^2}} \\ &= \frac{50 + 96.75}{2} - \sqrt{\frac{1}{4}\left(50 - 96.75\right)^2 + \frac{\left(-16.43\right)^2}{1.1^2}} \\ &= 45.64\end{aligned}$$

$$\begin{aligned}\omega_{n2}^2 &= \frac{A_1 + A_3}{2} + \sqrt{\frac{1}{4}\left(A_1 - A_3\right)^2 + \frac{A_2^2}{r_y^2}} \\ &= \frac{50 + 96.75}{2} + \sqrt{\frac{1}{4}\left(50 - 96.75\right)^2 + \frac{\left(-16.43\right)^2}{1.1^2}} \\ &= 101.11\end{aligned}$$

According to Eq. 4.92 in the textbook, the bounce oscillation center, L_{c1}, is

$$L_{c1} = \frac{A_2}{\omega_{n1}^2 - A_1} = \frac{-16.43}{45.64 - 50} = 3.77 \text{ m}$$

According to Eq. 4.93 in the textbook, the pitch oscillation center, L_{c2}, is

$$L_{c2} = \frac{A_2}{\omega_{n2}^2 - A_1} = \frac{-16.43}{101.11 - 50} = -0.32 \text{ m}$$

The bounce oscillation center is on the rear and is 3.77 meters from the center of gravity (CG) of the vehicle. The pitch oscillation center is within the wheelbase and is 0.32 meter in front of the vehicle CG.

Problem 4.13

A vehicle can be simplified as a five-degrees-of-freedom model, as shown in Figure P4.6. Establish the dynamic equation of the model.

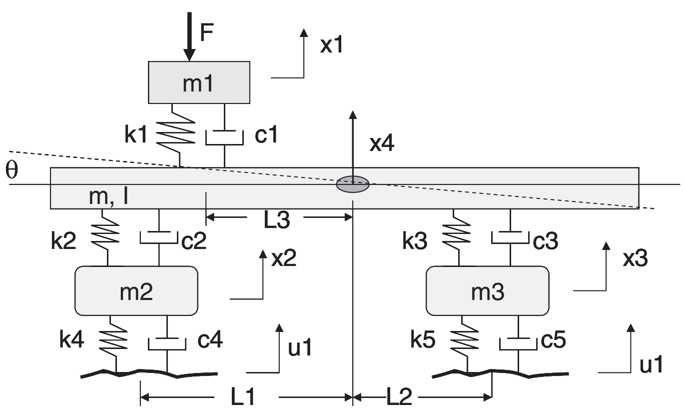

Figure P4.6 A five-degrees-of-freedom simplified vehicle model.

Solution to Problem 4.13

The vehicle has five degrees of freedom. The dynamic equation can be established by the Newton method or the Lagrange method. According to the Newton method, the dynamic equations are

$$m_1\ddot{x}_1 = -c_1\left[\dot{x}_1 - \left(\dot{x}_4 + l_3\dot{\theta}\right)\right] - k_1\left[x_1 - \left(x_4 + l_3\theta\right)\right]$$

$$m_2\ddot{x}_2 = -c_2\left[\dot{x}_2 - \left(\dot{x}_4 + l_1\dot{\theta}\right)\right] - k_2\left[x_2 - \left(x_4 + l_1\theta\right)\right] - c_4\left(\dot{x}_2 - \dot{u}_1\right) - k_4\left(x_2 - u_1\right)$$

$$m_3\ddot{x}_3 = -c_3\left[\dot{x}_3 - \left(\dot{x}_3 - l_2\dot{\theta}\right)\right] - k_3\left[x_3 - \left(x_4 - l_2\theta\right)\right] - c_5\left(\dot{x}_3 - \dot{u}_1\right) - k_5\left(x_3 - u_1\right)$$

$$\begin{aligned} m_4\ddot{x}_4 = &- c_1\left[\left(\dot{x}_4 + l_3\dot{\theta}\right) - \dot{x}_1\right] - k_1\left[\left(x_4 + l_3\theta\right) - x_1\right] \\ &- c_2\left[\left(\dot{x}_4 + l_1\dot{\theta}\right) - \dot{x}_2\right] - k_2\left[\left(x_4 + l_1\theta\right) - x_2\right] \\ &- c_3\left[\left(\dot{x}_4 - l_2\dot{\theta}\right) - \dot{x}_3\right] - k_3\left[\left(x_4 - l_2\theta\right) - x_3\right] \end{aligned}$$

$$
\begin{aligned}
I\ddot{\theta} = &\left\{c_1\left[\left(\dot{x}_4 + l_3\dot{\theta}\right) - \dot{x}\right]_1 + k_1\left[\left(x_4 + l_3\theta\right) - x_1\right]\right\} l_3 \\
&- \left\{c_2\left[\left(\dot{x}_4 + l_1\dot{\theta}\right) - \dot{x}_2\right] + k_2\left[\left(x_4 + l_1\theta\right) - x_2\right]\right\} l_1 \\
&- \left\{c_3\left[\left(\dot{x}_4 - l_2\dot{\theta}\right) - \dot{x}_3\right] + k_3\left[\left(x_4 - l_2\theta\right) - x_3\right]\right\} l_2
\end{aligned}
$$

The above equation can be written in matrix format. Then the modal frequencies and the modal shapes can be obtained.

Problem 4.14

Describe the methods used to evaluate ride quality, and compare their advantages and disadvantages.

Solution to Problem 4.14

Ride quality evaluation can be divided into subjective evaluation and objective evaluation. In objective evaluation, two major methods are used: (1) single value evaluation, and (2) spectrum evaluation. Figure P4.7 shows the categories.

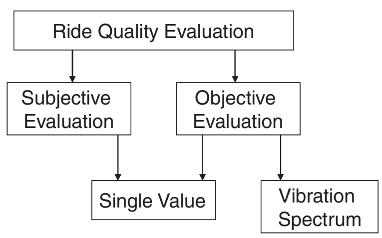

Figure P4.7 Categories of ride quality evaluation.

With subjective evaluation, trained people rate the ride quality according to their driving experience. The vehicle evaluation rating (VER) is divided into 10 levels, as shown in Figure P4.8. The VER number "1" means that the ride quality is very poor and that all passengers find it unacceptable. The number "10" means that the ride quality is excellent and that all passengers cannot feel vibrations in the vehicle. Among the 10 numbers, the higher numbers indicate better ride quality. Subjective evaluation is simple and fast and relates directly to passengers. However, the rating varies from passenger to passenger and from time to time.

Objective evaluation means that the seats' and/or the passengers' vibrations are recorded by testing or simulation. The data will be used for ride quality evaluation. Two methods are used in objective evaluation: (1) single value, and (2) vibration spectrum. The single value method means that ride quality is evaluated by a single value, including SEAT (Seat Effective Amplitude Transmissibility), RMS (root mean square), and VDV (vibration dose value). Vibration magnitude and frequency are considered and then are weighted in a single value. SEAT is used to evaluate seat ride quality but not to directly evaluate human body ride quality. RMS and VDV values can be used to evaluate human body ride quality. The RMS value is good for smooth vibration but is not suitable for road profiles with high crest factors. The VDV value is a good evaluation method for high crest factor vibration. The single value method is simple and instinctive for ride quality evaluation. However, the vibration at each frequency range cannot be obtained.

To understand vibration over the full range of frequency, the spectrum method must be used. Usually, the acceleration spectrum and the velocity spectrum are used. For a given number of engine revolutions per minute, the spectrum depends on the frequency, but velocity is used more often than acceleration.

<table>
<tr><td></td><td colspan="4">UNACCEPTABLE</td><td colspan="2">BORDERLINE ACCEPTABLE</td><td colspan="4">ACCEPTABLE</td></tr>
<tr><td>RG Rating Scale</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>Refinement</td><td colspan="3">Not acceptable</td><td>Objection-able</td><td>Requires improve-ment</td><td>Medium</td><td>Light</td><td>Very light</td><td>Trace</td><td>Not noticeable</td></tr>
<tr><td>Condition Noted By:</td><td>All customers</td><td colspan="3">Average customer</td><td colspan="3">Critical customer</td><td colspan="2">Trained observer</td><td>Not perceptible</td></tr>
</table>

Figure P4.8 Vehicle evaluation rating (VER).

Problem 4.15

If the SEAT values are 30% and 130%, what are the meanings of the two values?

Solution to Problem 4.15

The SEAT value is a single value used to evaluate the quality of a seat ride. The seat response spectrum, the seat track spectrum, and a weighted frequency function are used to calculate the SEAT value. The 30% SEAT value means that the vibration on the seat butt is only 30% of that on the seat track. Thus, the vibration is reduced through the seat. The 130% value means that the vibration on the seat butt is 130% of that on the seat track, which means that the vibration is amplified through the seat.

Problem 4.16

Describe the differences among passive control, semi-active control, and active control. Use the engine mount to describe the problem.

Solution to Problem 4.16

Passive control use properties of structure and materials. No extra energy and mechanisms are used in this type of control. For example, a rubber mount is a passive mount. The mount uses only the elastic properties of the material and structure to achieve the goal of vibration control.

Active control uses extra energy and control mechanisms in addition to the passive control structure. For example, Figure P4.9 shows an active engine mount. This mount consists of three parts: (1) a traditional hydraulic mount (passive mount), (2) an electromagnetic actuator, and (3) a load sensor. The load sensor measures system response and provides signal to a control system. The control system sends a command to the actuator that provides control force to the system.

A semi-active mount uses a control system based on a passive mount. Usually, the control system provides a current or a magnetic field to the liquid inside the mount to change the properties of the mount. But no extra energy system is found in a semi-active mount.

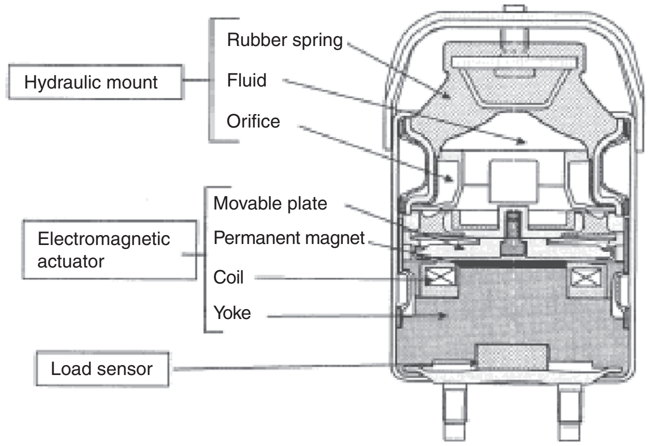

Figure P4.9 Active engine mount.

Problem 4.17

Both the RMS value and the VDV value can be used to describe ride quality. Describe their common traits and differences.

Solution to Problem 4.17

The RMS, which is the root mean square, is defined as

$$\bar{a}_w = \left[\frac{1}{T}\int_0^T a_w^2(t)\,dt\right]^{1/2}$$

The VDV, which is the vibration dose value, is defined as

$$VDV = \left[\int_{t=0}^{t=T} a_w^4(t)\,dt\right]^{1/4}$$

The commonalities between the RMS and the VDV are that both use a single value for ride quality evaluation, and both measured accelerations are weighted.

Their differences are that the RMS averages acceleration, whereas the VDV does not. Thus, the RMS can be used for smooth vibration data but cannot be used for cases in which the crest factors are higher than 9. Because the VDV is not averaged, the value can be used for cases with high crest factors.

Problem 4.18

At a certain level of engine revolutions per minute, the seat track vibration varies with the frequency. The seat track vibration is measured easily by an accelerometer. Why is velocity usually used to evaluate seat track vibration, instead of acceleration?

Solution to Problem 4.18

From human body sensitivity to the vibration curves in Figure 4.44, it is known that the most highly sensitive frequency range is from 4 to 8 Hz. Above 8 Hz, the sensitivity is an upward straight line (i.e., the sensitivity increases as the frequency increases). If the

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acceleration is integrated in the frequency domain, the corresponding velocity can be obtained as

$$v(f) = \int a(f)\,df = \frac{a(f)}{j2\pi f}$$

The acceleration and velocity curves are plotted in Figure P4.10. Above 8 Hz, the velocity is a flat straight line.

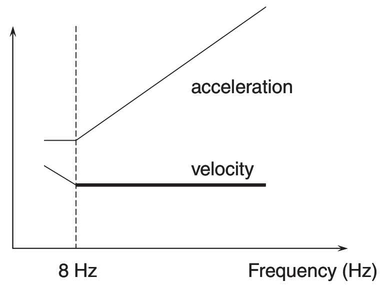

Figure P4.10 Acceleration and velocity curves.

In vehicle vibration analysis, the concerned vibration frequency range reaches up to 100 Hz. The frequencies of most vibration sources are higher than 8 Hz. After velocity is used, the velocity can be compared by different flat straight lines. This is much more convenient for comparing the level of vehicle vibration and is the reason why velocity usually is used.

Chapter 5

Vehicle Rollover Analysis

Problem 5.1

What are the reasons that cause vehicle rollover to occur?

Solution to Problem 5.1

By definition, rollover occurs as a vehicle rotates 90° or more around its longitudinal axis. If the center of gravity (CG) of an inclined vehicle exceeds its stable point, rollover will occur. Three factors influence rollover: (1) the vehicle, (2) the environment, and (3) the occupant.

Vehicle structure and parameters are very important in rollover. The CG height of the vehicle and the tire tread width determine the steady static rollover threshold. A flexible tire and suspension reduce the rollover threshold. Many other vehicle factors influence rollover stability.

The environment refers to the condition of the road. Rollover can occur on any kind of road, such as a smooth and flat road, a rough road, or a cross-slope road. Rollover can be divided into two categories: (1) tripped and (2) untripped. A tripped rollover is caused by a vehicle hitting against an obstacle. An untripped rollover is caused by excessive lateral acceleration.

The occupant refers to drivers who have bad driving habits, such as severe steering or falling sleep at the wheel.

In vehicle rollover analysis and design, the reasons that rollover occurs are focused on vehicle parameters and structures.

Problem 5.2

Why is rollover analysis important? What are the major research fields of rollover?

Solution to Problem 5.2

Rollover is involved in only a small percentage of the overall number of vehicle accidents. However, rollover occupies a higher percentage of fatalities and incapacity in accidents. Rollover can eject occupants from a vehicle through a window and can cause occupants to impact the interior structure of the vehicle. Hence, it is very important to understand the mechanisms of rollover, vehicle structure, safety prevention systems, and so forth.

Rollover research focuses on three fields: (1) the mechanism of vehicle rollover, (2) the detection and prevention of rollover, and (3) occupant protection. Research on vehicle rollover mechanisms includes analyzing the vehicle parameters during rollover, building and analyzing rollover models, and so forth. Detection and prevention of rollover include installation of efficient and prompt rollover detection sensors and provision of a rollover control system. Occupant protection includes analyzing the occupants' rollover models, rollover crash scenarios, head excursions, and time histories.

Problem 5.3

How many metrics are needed to describe the steady static state rollover? Please explain the differences among these metrics.

Solution to Problem 5.3

The steady static state means that the roll velocity and acceleration are neglected and only the lateral acceleration is considered at rollover incipient. Three metrics typically are used to describe a steady-state rollover, as follows:

- Static stability factor (SSF)
- Tilt table ratio (TTR)
- Side pull ratio (SPR)

The static stability factor (SSF) is defined as a nondimensional acceleration, $\frac{a_y}{g} = \frac{d}{2h}$, as the rollover threshold. The rollover threshold depends on only two basic vehicle parameters: the CG height, h, and the tread width, d. This method is simple and provides a useful metric by which to study rollover propensity. However, the metric overestimates the actual static rollover threshold of real vehicles because many factors are neglected, such as the deflection of tires and the suspension.

The tilt table ratio (TTR) is based on putting a vehicle on a slope table with slope angle ϕ and is expressed as

$$\mathrm{TTR} = \frac{a_s}{g} = \tan\phi$$

The TTR depends on only the table slope angle; hence the method is simple and safe. However, this method cannot provide a real vehicle rollover threshold that can be used only for comparison of rollover propensity among vehicles.

Similar to the TTR, the side pull ratio (SPR) is based on putting a vehicle on a flat table and is expressed as

$$\mathrm{SPR} = \frac{F_p}{mg}$$

The SPR uses the proper magnitude of vertical weight and lateral load applied on the vehicle; hence, it is more accurate than the SSF and TTR at the beginning of rollover. However, the pull force is concentrated on a single point, and the CG height of the vehicle always changes.

Problem 5.4

A vehicle weighs 6000 kg, and the tire radial stiffness is 400,000 N/m. At the moment when rollover occurs, what is the tire deflection of the inside tire and the outside tire?

Solution to Problem 5.4

At the moment rollover occurs, the inside tire loses contact with the ground surface. Therefore, that tire is not subjected to vehicle load. The deflection of the inside tire is zero. All the weight of the vehicle is placed on the outside tire. The deflection of the outside tire is

$$\Delta_{\mathrm{outside}} = \frac{F_{\mathrm{outside}}}{K} = \frac{mg}{K} = \frac{6000 * 9.8}{400000} = 0.147\ \mathrm{m}$$

Problem 5.5

A vehicle weighs 3500 kg. The CG height of the vehicle is 1.0 m, and the tread width is 1.9 m. If the rotational angle caused by the tire deflection of the vehicle at the moment rollover occurs is 8°, calculate the radial stiffness of the tire.

Solution to Problem 5.5

The rotational angle at rollover incipient caused by tire deflection is

$$\theta_1 = \frac{\Delta r}{d/2} = \frac{mg}{dK_{\mathrm{tire}}}$$

Thus, the stiffness of the tire is

$$K_{\mathrm{tire}} = \frac{mg}{d\theta_1} = \frac{3500 * 9.8}{1.9 * 8 * \dfrac{3.14}{180}} \approx 129{,}000\ \mathrm{N/m}$$

Problem 5.6

A truck weighs 15,000 kg. The roll center height of the truck is 0.6 m, and the CG height of the body is 1.7 m. Determine the minimal suspension roll stiffness.

Solution to Problem 5.6

According to Eq. 5.21 in the textbook, the minimal linear stiffness of each suspension spring is

$$K = \frac{2mgh_2}{b^2}$$

The suspension roll stiffness is

$$K_{\theta_2} = \frac{b^2}{2}K$$

Substitute K into the above equation to obtain

$$K_{\theta_2} = mgh_2 = 15{,}000 * 9.8 * \left(1.7 - 0.6\right) = 161{,}700 \text{ Nm/rad}$$

Problem 5.7

A vehicle weighs 16,000 kg. The tread width of the tires is 2.2 m, and the CG height of the body is h = 1.8 m. The suspension is assumed to be a rigid body. The radial stiffness of one tire is 1,000,000 N/m. Calculate the lateral acceleration and the rotation angle when rollover occurs.

Solution to Problem 5.7

The tire dimensional stiffness is

$$K^*_{tire} = \frac{K_{tire}}{mg/d} = \frac{1{,}000{,}000}{16{,}000 * 9.8/2.2} = 14.03$$

The rotational angle is

$$\theta_1^R = \frac{1}{K^*_{tire}} = \frac{1}{14.03} * \frac{180}{3.14} = 4.1^0$$

The lateral acceleration is

$$\begin{aligned} a_y &= \left(\frac{d}{2h} - \theta_1^R\right)g \\ &= \left(\frac{2.2}{2 * 1.8} - \frac{1}{14.03}\right) * 9.8 \\ &= 5.40 \text{ m/s}^2 \end{aligned}$$

Problem 5.8

A vehicle weighs 16,000 kg. The tread width of the tires is 2.2 m. The roll center height is 0.8 m, and the CG height of the body is 1.8 m. The distance between the two suspension springs is 1.9 m. Assume that the tire is rigid. To avoid a rotational angle caused by a suspension roll moment larger than 15°, calculate the minimal stiffness of the suspension. Also calculate the lateral acceleration when rollover occurs.

Solution to Problem 5.8

By Eq. 5.13 in the textbook, the rollover threshold is

$$\frac{a_y}{g} = \frac{d}{2h} - \theta_1 - \frac{h_2}{h}\theta_2$$

The rollover threshold also can be expressed as Eq. 5.28 as

$$\frac{a_y}{g} = \frac{d}{2h} - \frac{1}{K^*_{tire}} - \frac{d}{2h}\frac{1}{1 + \frac{h}{h_2}\left(K^*_{\theta_2} - 1\right)}$$

When the two previous equations are compared, the rotational angle of the body is

$$\theta_2 = \frac{d}{2h_2}\frac{1}{1 + \frac{h}{h_2}\left(K^*_{\theta_2} - 1\right)}$$

Therefore, the minimal dimension-less roll stiffness is

$$\begin{aligned} K^*_{\theta_2} &= \frac{d}{2h\theta_2} + \frac{h_1}{h} \\ &= \frac{2.2}{2 * 1.8 * 15 * \left(3.14/180\right)} + \frac{0.8}{1.8} \\ &= 2.78 \end{aligned}$$

The corresponding roll stiffness is

$$\begin{aligned} K_{\theta_2} &= mgh_2K^*_{\theta_2} \\ &= 16{,}000 * 9.8 * \left(1.8 - 0.8\right) * 2.78 \\ &\approx 436{,}000 \text{ Nm/rad} \end{aligned}$$

The suspension stiffness is

$$\begin{aligned} K &= \frac{2}{b^2}K_{\theta_2} \\ &= \frac{2}{1.9^2} * 436{,}000 \\ &\approx 241{,}000 \text{ N/m} \end{aligned}$$

The lateral acceleration when rollover occurs is

$$a_y = \left[\frac{d}{2h} - \frac{d}{2h}\frac{1}{1+\frac{h}{h_2}\left(K^*_{\theta_2} - 1\right)}\right] g$$

$$= \left[\frac{2.2}{2*1.8} - \frac{2.2}{2*1.8} * \frac{1}{1+\frac{1.8}{1}\left(2.78 - 1\right)}\right] g$$

$$= 0.466\ g$$

Problem 5.9

A vehicle weighs 2000 kg. The tread width of the tires is 2 m. The roll center is 0.4 m, and the CG height of the body is 1.0 m. Assume that the tire is rigid. The distance between the two suspension springs is 1.8 m, and the spring stiffness is 30,000 N/m. Calculate the body roll angle caused by the flexible suspension.

Solution to Problem 5.9

The roll stiffness of body suspension is

$$K_{\theta_2} = \frac{b^2}{2}K$$

$$= \frac{1.8^2}{2} * 30{,}000$$

$$= 48{,}600\ \text{Nm/rad}$$

The dimension-less roll stiffness is

$$K^*_{\theta_2} = \frac{K_{\theta_2}}{mgh_2}$$

$$= \frac{48{,}600}{2000 * 9.8 * \left(1.0 - 0.4\right)}$$

$$= 4.13$$

According to Problem 5.8, the rotational angle of the body is

$$\theta_2 = \frac{d}{2h_2}\frac{1}{1+\frac{h}{h_2}\left(K^*_{\theta_2} - 1\right)}$$

$$= \frac{2}{2*\left(1.0 - 0.4\right)} * \frac{1}{1+\frac{1.0}{1.0-0.4} * \left(4.13 - 1\right)}$$

$$= 0.27\ \text{rad},\ \text{or}\ 15.5°$$

Problem 5.10

Compare the degrading contribution of a flexible tire and a flexible suspension to the rollover threshold. The vehicle weighs 8000 kg, and the tread width of the tire is 2.2 m. The roll center height is 0.6 m, and the CG height of the body is 1.4 m. The distance between the two suspension springs is 1.9 m. The tire stiffness is 600,000 N/m, and the suspension stiffness is 70,000 N/m.

Solution to Problem 5.10

The steady static rollover threshold is

$$\left(\frac{a_y}{g}\right)_s = \frac{d}{2h} = \frac{2.2}{2 * 1.4} = 0.786$$

The rollover threshold for a flexible tire and suspension is

$$\frac{a_y}{g} = \frac{d}{2h} - \theta_1^R - \theta_2^R$$

Here, θ_1^R is the rollover threshold degrading that is caused by the flexibility of the tire:

$$\begin{aligned}
\theta_1^R &= \frac{\Delta r}{d/2} \\
&= \frac{mg}{dK_{tire}} \\
&= \frac{8,000 * 9.8}{2.2 * 600,000} \\
&= 0.059 \text{ rad}, \quad \text{or} \quad 3.38°
\end{aligned}$$

The suspension roll dimension-less stiffness is

$$\begin{aligned}
K_{\theta_2}^* &= \frac{K_{\theta_2}}{mgh_2} \\
&= \frac{Kb^2/2}{mgh_2} \\
&= \frac{70,000 * 1.9^2/2}{8,000 * 9.8 * (1.4 - 0.6)} \\
&= 2.02
\end{aligned}$$

Here, θ_2^R is the rollover threshold degrading caused by the flexibility of the suspension:

$$\begin{aligned}
\theta_2^R &= \frac{d}{2h}\frac{1}{1 + \frac{h}{h_2}\left(K_{\theta_2}^* - 1\right)} \\
&= \frac{2.2}{2 * 1.4}\frac{1}{1 + \frac{1.4}{1.4 - 0.6}(2.02 - 1)} \\
&= 0.282 \text{ rad}, \quad \text{or} \quad 16.16°
\end{aligned}$$

The normalized rollover threshold is

$$\left(\frac{a_y}{g}\right)_s = 0.786 - 0.059 - 0.282 = 0.445 \text{ rad}$$

Figure P5.1 shows the influence of the tire stiffness and the suspension stiffness on the rollover threshold.

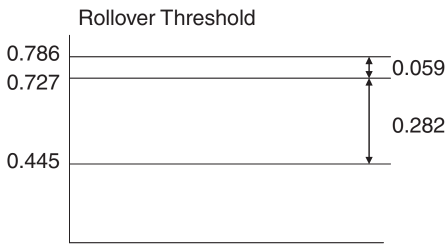

Figure P5.1 Influence of tire stiffness and suspension stiffness on rollover threshold.

The flexible tire and suspension reduce the rollover threshold of a rigid body by 7.5% and 35.9%, respectively. In this problem, the suspension degrades the rollover threshold to a greater extent than the tire does.

Problem 5.11

A vehicle weighs 10,000 kg, and the tread width of the tires is 2.3 m. The roll center height is 0.7 m, and the CG height of the body is 1.5 m. The distance between the two suspension springs is 1.9 m. To control the rollover threshold within 65% of the static threshold, determine the tire and suspension stiffness.

Solution to Problem 5.11

The rollover threshold for a flexible tire and suspension is

$$\frac{a_y}{g} = \frac{d}{2h} - \theta_1^R - \theta_2^R$$

This problem states that the threshold should be 65% of that of the rigid body, that is,

$$\frac{a_y}{g} = \frac{d}{2h} - \theta_1^R - \theta_2^R = 0.65 * \frac{d}{2h}$$

Thus,

$$\begin{aligned}\theta_1^R + \theta_2^R &= 0.35 * \frac{d}{2h} \\ &= 0.35 * \frac{2.3}{2 * 1.5} \\ &= 0.27 \text{ rad}\end{aligned}$$

From Chapter 4 titled "Ride Dynamics," we know that the tire stiffness is much greater than that of the suspension, if it is assumed that $K_{tire} \approx 9K$.

Here, θ_1^R, the rollover threshold degrading caused by the flexibility of the tire, is

$$\theta_1^R = \frac{\Delta r}{d/2} = \frac{mg}{dK_{tire}} = \frac{10,000 * 9.8}{2.3K_{tire}} = \frac{42,608}{K_{tire}}$$

Substituting

$$K_{\theta_2}^* = \frac{K_{\theta_2}}{mgh_2} = \frac{Kb^2/2}{mgh_2}$$

into

$$\theta_2^R = \frac{d}{2h}\frac{1}{1 + \frac{h}{h_2}\left(K_{\theta_2}^* - 1\right)}$$

gives

$$\begin{aligned}
\theta_2^R &= \frac{d}{2h}\frac{1}{1 + \frac{h}{h_2}\left(\frac{Kb^2/2}{mgh_2} - 1\right)} \\
&= \frac{2.3}{2 * 1.5}\frac{1}{1 + \frac{1.5}{1.5 - 0.7}\left(\frac{K * 1.9^2/2}{10,000 * 9.8 * (1.5 - 0.7)} - 1\right)} \\
&= \frac{0.77}{4.3 * 10^{-5}K - 0.875}
\end{aligned}$$

Therefore, we have

$$\theta_1^R + \theta_2^R = \frac{42,608}{K_{tire}} + \frac{0.77}{4.3 * 10^{-5}K - 0.875} = 0.27$$

Substituting $K_{tire} = 9K$ into the previous equation gives

$$\frac{42,608}{9K} + \frac{0.77}{4.3 * 10^{-5}K - 0.875} = 0.27$$

The suspension stiffness is $K \approx 100,000$ N/mm.

The tire stiffness is $K_{tire} = 9\ K = 900,000\ N \cdot m$.

Problem 5.12

Compare the differences between the static rollover threshold and the dynamic rollover threshold.

Solution to Problem 5.12

Three steady static rollover thresholds are known: safety stability factor (SSF), tilt table ratio (TTR), and side pull ratio (SPR). For TTR and SPR, the vehicle is placed on tables. Rollover occurs by raising the table or by pulling the vehicle. The vehicle is in a “static” state. For SSF, rollover is caused by excessive lateral acceleration, and the vehicle is in a steady static state.

The dynamic rollover threshold can be described in four ways: dynamic stability index (DSI), rollover prevention energy reserve (RPER), rollover prevention metric (RPM), and critical sliding velocity (CSV). Included in these dynamic thresholds are rotational acceleration/velocity, lateral acceleration, kinetic energy, and potential energies.

In comparison of SSF

$$\mathrm{SSF} = \frac{a_y}{g} = \frac{d}{2h}$$

with DSI,

$$\mathrm{DSI} = \frac{a_y}{g} + \frac{I_b\ddot{\varphi}}{mgh}$$

it can be seen that DSI has one more term than SSF (i.e., $\frac{I_b\ddot{\varphi}}{mgh}$). In this term, rotational acceleration is included. The DSI also includes the lateral acceleration term and the rotational acceleration term; therefore, the lateral acceleration threshold of the DSI will be lower than that of the SSF. The SSF overestimates the rollover threshold, but the DSI is close to actual cases.

Problem 5.13

A vehicle is simplified as a box shown in Figure P5.2. The parameters are as follows: a = 1.6 m, d = 1.8 m, and the mass is 1500 kg. The center of gravity of the vehicle is located at O. Calculate the critical sliding velocity (CSV).

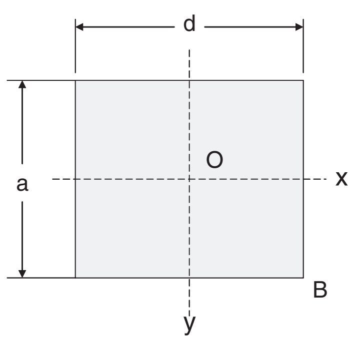

Figure P5.2 A vehicle simplified as a box.

Solution to Problem 5.13

When the vehicle travels and impacts a hinge at point B, the linear kinetic energy will be transferred into the rotational kinetic energy. The moment of inertia around point B should be

$$
\begin{aligned}
I_B &= \sqrt{I_x^2 + I_y^2} + m\left(\frac{a^2}{4} + \frac{d^2}{4}\right) \\
&= \frac{m}{12}\sqrt{a^4 + d^4} + m\left(\frac{a^2}{4} + \frac{d^2}{4}\right) \\
&= \frac{1500}{12}\sqrt{1.8^4 + 1.6^4} + 1500 * \left(\frac{1.6^2}{4} + \frac{1.8^2}{4}\right) \\
&= 516 + 2175 \\
&= 2691\ \mathrm{kg} - \mathrm{m}^2
\end{aligned}
$$

According to Eq. 5.52 in the textbook and with $h = d/2$, the critical sliding velocity is

$$
\begin{aligned}
V &= \sqrt{\frac{2gI_B}{mh}\left(\sqrt{1 + \left(\frac{d}{2h}\right)^2} - 1\right)} \\
&= \sqrt{\frac{2 * 9.8 * 2691}{1500 * 0.8}\left(\sqrt{1 + \left(\frac{1.8}{2 * 0.8}\right)^2} - 1\right)} \\
&= 4.71\ \mathrm{m/s}
\end{aligned}
$$

Problem 5.14

Based on Problem 5.13, calculate the initial rotational velocity.

Solution to Problem 5.14

At the moment of impact, the momentum is conservative, that is,

$$mVh = I_B\dot{\varphi}_0$$

Hence, the initial rotational velocity is

$$\dot{\varphi}_0 = \frac{mVh}{I_B} = \frac{1500 * 4.71 * 0.8}{2691} = 2.1\ \mathrm{rad/s}$$

Problem 5.15

Describe the passengers' injury scenario in rollover accidents. Also describe the purpose of rollover simulation and testing.

Solution to Problem 5.15

Fatalities and incapacity caused by rollover are found in a high percentage of vehicle accidents. Rollover injuries include two possibilities. One is that the occupants are ejected to the outside of the vehicle via windows because those occupants did not use their seatbelts. The seatbelt is one of the most important factors in saving lives in rollover accidents. The other is that the occupants who used their seatbelts are injured by coming into contact with interior structures of the vehicle.

Occupant safety can be analyzed by testing and by computer-aided engineering (CAE) simulation. The testing can be processed in test fixtures. Both dummies and human volunteers are used for the testing. However, for testing with a high rollover rate, only dummies can be used. Occupants' kinematics were measured by rotating the test fixtures, and their motion can be recorded by cameras.

Computer-aided engineering simulation is a fast and cost-saving way to analyze rollover. Simulation models usually include both occupants and vehicle structures. Two kinds of models are known: finite element models, and multi-body facet models. The MADYMO (MAthematical DYnamic MOdels) model is the most popular rollover model. Occupants' kinematics, contact forces, and many other parameters can be calculated by using these models.

Problem 5.16

Describe how an active rollover preventive system works.

Solution to Problem 5.16

A rollover preventive system includes two major parts: (1) a detection system, and (2) an anti-rollover control system, as shown in Figure P5.3. The functions of a sensing system are to accurately estimate the roll rate of the vehicle, the roll angle, the roll velocity and acceleration, and the lateral and vertical acceleration, as well as to provide timely and accurate command of the rollover control system. Detectors used in a detection system identify signals that are compared with a threshold. If the detected signals are close to the threshold, the signals will trigger a control system that provides anti-rollover control to the vehicle.

Figure P5.3 Parts of a rollover preventive system.

Chapter 6

Handling Dynamics

Problem 6.1

Use Newton's laws to derive a mathematical model of a vehicle similar to the bicycle model to describe the motion in the lateral direction. Simplify the model to be a steady-state model, which means that the steering wheel angle is a constant.

Solution to Problem 6.1

Define yaw velocity as r and the sideslip angle as β (see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7). Denote that F_{yf} is the lateral force at the front axle, and F_{yr} is the lateral force at the rear axle, v is the forward velocity, R is the radius of the turn, δ is the steer angle at the front wheels (degrees), L is the wheelbase (meters), and Vx and Vy are the longitudinal and lateral components of the center of gravity (CG) velocity. If a constant longitudinal velocity of Vx = v is assumed, I is the moment of inertia of the vehicle about its yaw axis, m is the vehicle mass, and a and b are the distances of the front and rear axles from the CG, L = a + b. If we apply Newton's second law (Eq. 1.44 in the book *Road Vehicle Dynamics*) at the center of gravity along the lateral directions, we obtain

$$F_{yr} + F_{yf}\cos\delta = mv^2\cos\beta/R - m\dot{v}\sin\beta \tag{6.1.1}$$

Considering the moment equilibrium at the CG, we obtain

$$F_{yf}a\cos\delta = I\dot{r} + F_{yr}b \tag{6.1.2}$$

and consider the relationship

$$\begin{aligned} F_{yf} &= C_f\alpha_f \\ F_{yr} &= C_r\alpha_r \end{aligned} \tag{6.1.3}$$

For small angles, after referring to Figure 6.7 in the book *Road Vehicle Dynamics* (see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7), we can write

$$
\begin{aligned}
\sin\beta &= \beta \\
\sin\delta &= \delta \\
\cos\alpha &= 1 \\
\cos\delta &= 1 \\
\alpha_f &\approx \beta + \delta - a\frac{r}{v} \\
\alpha_r &\approx \beta + b\frac{r}{v}
\end{aligned}
\tag{6.1.4}
$$

By substituting the previous equations into Eq. 6.1.3, we get

$$F_{yf} = C_f\left(\beta + \delta - a\frac{r}{v}\right) \tag{6.1.5}$$

$$F_{yr} = C_r\left(\beta + b\frac{r}{v}\right) \tag{6.1.6}$$

By noting that $\dot{\gamma} = v/R$, we obtain the following relationship among yaw angle, forward velocity, and sideslip angle:

$$
\begin{aligned}
\psi &= \beta + \gamma \\
r &= \dot{\beta} + \dot{\gamma}
\end{aligned}
\tag{6.1.7}
$$

Substituting Eqs. 6.1.5 through 6.1.7 into Eqs. 6.1.1 and 6.1.2 gives

$$
\begin{aligned}
&C_r\left(\beta + b\frac{r}{v}\right) + C_f\left(\beta + \delta - a\frac{r}{v}\right) + mv\left(\dot{\beta} - r\right) + m\dot{v}\beta = 0 \\
&C_f a\left(\beta + \delta - a\frac{r}{v}\right) - I\dot{r} - C_r b\left(\beta + b\frac{r}{v}\right) = 0
\end{aligned}
\tag{6.1.8}
$$

The previous set of equations is nonlinear. By assuming the forward velocity to be constant (namely, $\dot{v} = 0$), we can rearrange these equations as

$$r\left(C_r\frac{b}{v} - C_f\frac{a}{v} - mv\right) + \dot{\beta}mv + \beta\left(C_f + C_r\right) + \delta C_f = 0 \tag{6.1.9a}$$

$$\dot{r}I + r\left(C_f\frac{a^2}{v} + C_r\frac{b^2}{v}\right) + \beta\left(C_r b - C_f a\right) - \delta a C_r = 0 \tag{6.1.9b}$$

If the forward velocity v, the cornering stiffness C_f and C_r, and parameters a, b, m, and I are known, with given steer angle δ, the sideslip angle β, yaw rate, or velocity r can be calculated.

$$\dot{\beta} = -\left(\frac{C_r + C_f}{mv}\right)\beta + \left(\frac{C_f b - C_r a}{mv^2} - 1\right)r + \frac{C_f}{mv}\delta \tag{6.1.10a}$$

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$$\dot{r} = \left(\frac{C_r b - C_f a}{I}\right)\beta - \left(\frac{C_r b^2 + C_f a^2}{I}\right) r + \frac{C_f}{I}\delta \tag{6.1.10b}$$

Or as a state equation, we have

$$P\dot{X} + QX = RU \tag{6.1.11}$$

where X is a state vector and U is the input vector.

$$\begin{aligned}
X &= \begin{bmatrix} \beta \\ r \end{bmatrix} \\
U &= \begin{bmatrix} \delta \end{bmatrix} \\
P &= \begin{bmatrix} m & \\ & I \end{bmatrix} \\
Q &= \begin{bmatrix} \dfrac{C_r + C_f}{v} & \dfrac{-C_r b + C_f a}{v} + v \\ \left(\dfrac{-C_r b + C_f a}{v}\right)\beta & \dfrac{+C_r b^2 + C_f a^2}{v} \end{bmatrix} \\
R &= \begin{bmatrix} \dfrac{C_f}{v} \\ \dfrac{C_f a}{I} \end{bmatrix}
\end{aligned} \tag{6.1.12}$$

By introducing the following parameters,

$$\begin{aligned}
a_{11} &= -\frac{C_r + C_f}{mv} \\
a_{12} &= -\left(\frac{C_r b - C_f a}{mv^2} - 1\right) \\
a_{21} &= -\frac{C_r b - C_f a}{I} \\
a_{22} &= -\frac{C_r b^2 + C_f a^2}{Iv} \\
b_1 &= \frac{C_f}{mv} \\
b_2 &= \frac{aC_f}{I}
\end{aligned} \tag{6.1.13}$$

Equation 6.1.10 can be expressed as

$$\begin{aligned}
\dot{\beta} &= a_{11}\beta + a_{12} r + b_1 \delta \\
\dot{r} &= a_{21}\beta + a_{22} r + b_2 \delta
\end{aligned} \tag{6.1.14}$$

By eliminating one variable β, we obtain

$$\ddot{r} - \left(a_{11} + a_{22}\right)\dot{r} + \left(a_{11}a_{22} - a_{12}a_{21}\right)r = b_2\dot{\delta} + \left(a_{21}b_1 - a_{11}b_2\right)\delta \qquad (6.1.15)$$

Similarly, we can get another equation,

$$\ddot{\beta} - \left(a_{11} + a_{22}\right)\dot{\beta} + \left(a_{11}a_{22} - a_{12}a_{21}\right)\beta = b_2\dot{\delta} + \left(a_{12}b_2 - a_{22}b_1\right)\delta \qquad (6.1.16)$$

$$\begin{aligned} \dot{\beta} &= a_{11}\beta + a_{12}r \\ \dot{r} &= a_{21}\beta + a_{22}r \end{aligned} \qquad (6.1.17)$$

Problem 6.2

Using Lagrange's equation of motion, derive the model in Problem 6.1, and compare its features with those in Problem 6.1 using Newton's law.

Solution to Problem 6.2

Based on Eq. 1.79 in the book *Road Vehicle Dynamics* (see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7), Lagrange's equation for a non-conservative system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q'_i \qquad (6.2.1)$$

where T and V are the kinetic and potential energy of the system, respectively, and Q' is a general force. For the system in Problem 6.1, the potential energy of the system is zero. The general coordinates can be selected as lateral displacement and yaw angle displacement

$$\begin{aligned} q_1 &= y \\ q_2 &= \psi \end{aligned} \qquad (6.2.2)$$

Note that

$$\begin{aligned} \beta &= \tan\beta = \dot{y}/v \\ r &= \dot{\psi} \end{aligned} \qquad (6.2.3)$$

Equation 6.2.1 can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = Q'_y \qquad (6.2.4)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = Q'_\psi \qquad (6.2.5)$$

The general forces Q'_y and Q'_ψ are the force and moment corresponding to y and ψ, respectively.

$$Q'_y = F_{yf} + F_{yr} \qquad (6.2.6)$$

$$Q'_{\psi} = F_{yf}a - F_{yr}b \tag{6.2.7}$$

Note that $L = T - V$ and $V = 0$. Therefore,

$$T = \frac{1}{2}m\left(v^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\psi}^2 \tag{6.2.8}$$

By substituting Eqs. 6.2.2, 6.2.3, and 6.2.5 through 6.2.7 into Eqs. 6.2.4 and 6.2.5, we get

$$F_{yr} + F_{yf}\cos\delta = mv^2\cos\beta/R - m\dot{v}\sin\beta \tag{6.2.9}$$

$$F_{yf}a\cos\delta = I\dot{r} + F_{yr}b \tag{6.2.10}$$

Equations 6.2.9 and 6.2.10 are similar to Eqs. 6.1.1 and 6.1.2 in Problem 6.1. Thus, the following derivation is the same as that of Eqs. 6.1.1 through 6.1.9 in Problem 6.1. Accordingly, the final obtained motion equations are identical.

Problem 6.3

Use the model obtained in Problem 6.1 to determine the understeer coefficient (parameter, gradient).

Solution to Problem 6.3

Based on Problem 6.1, we obtain Eq. 6.1.14 in Problem 6.1 by eliminating one variable β. Thus,

$$\ddot{r} - \left(a_{11} + a_{22}\right)\dot{r} + \left(a_{11}a_{22} - a_{12}a_{21}\right)r = b_2\dot{\delta} + \left(a_{21}b_1 - a_{11}b_2\right)\delta \tag{6.3.1}$$

Similarly, we can get another equation,

$$\ddot{\beta} - \left(a_{11} + a_{22}\right)\dot{\beta} + \left(a_{11}a_{22} - a_{12}a_{21}\right)\beta = b_2\dot{\delta} + \left(a_{12}b_2 - a_{22}b_1\right)\delta \tag{6.3.2}$$

Consider steady cornering when a vehicle drives through the curve at low lateral acceleration and low lateral forces are needed. At the wheels, hardly any lateral slip occurs. In the ideal case, with a vanishing lateral slip, the wheels move only in a circumferential direction. Then, $\ddot{\beta} = 0$, $\dot{\beta} = 0$, $\dot{\delta} = 0$, and $\delta =$ constant. Equation 6.3.2 becomes

$$\left(a_{11}a_{22} - a_{12}a_{21}\right)\beta = \left(a_{12}b_2 - a_{22}b_1\right)\delta \tag{6.3.3}$$

Then, we have

$$\frac{\beta}{\delta} = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{-C_fC_rb\left(a + b\right) + C_famv^2}{C_fC_r\left(a + b\right)^2 + \left(C_rb - C_fa\right)mv^2} \tag{6.3.4}$$

Similarly,

$$\begin{aligned}\frac{r}{\delta} &= \frac{a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}} \\ &= \frac{v\left(a + b\right)C_fC_r}{\left(a + b\right)^2C_fC_r + mv^2\left(bC_r - aC_f\right)}\end{aligned} \tag{6.3.5}$$

Equation 6.3.5 is defined as yaw velocity gain. We define understeer gradient as

$$K = \frac{m\left(bC_r - aC_f\right)}{L^2 C_f C_r} \tag{6.3.6}$$

with the unit degrees per gram (deg/g) or radians per meter per second squared (rad/m/s^2).

Then we have

$$\frac{r}{\delta} = \frac{v/L}{1 + v^2 K} \tag{6.3.7}$$

Problem 6.4

Develop a mathematical model for the transient handling analysis of a passenger car under the effects of lateral wind gust. The model should reflect the speeds and positions for a vehicle moving on a horizontal ground plane. The speed in the longitudinal direction can be assumed to be constant. Propose quantities that are state variables and control variables.

Solution to Problem 6.4

Consider the bicycle model presented in the book *Road Vehicle Dynamics* (see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7). Define the yaw velocity as $r = \dot{\psi}$ and the sideslip angle as β. Assume that F_w is the wind gust excitation force and that M_w is the wind gust excitation moment applied on the vehicle. By applying Newton's second law at the center of gravity along the lateral directions, respectively, we obtain

$$F_{yr} + F_{yf}\cos\delta - F_w = mv^2\cos\beta/R - m\dot{v}\sin\beta \tag{6.4.1}$$

Considering the moment equilibrium at the center of gravity, we obtain

$$F_{yf}a\cos\delta + M_w = I\dot{r} + F_{yr}b \tag{6.4.2}$$

Consider the relationship

$$\begin{aligned} F_{yf} &= C_f\alpha_f \\ F_{yr} &= C_r\alpha_r \end{aligned} \tag{6.4.3}$$

For small angles, we can write

$$\begin{aligned} \alpha_f &\approx \beta + \delta - a\frac{r}{v} \\ \sin\beta &= \beta \\ \sin\delta &= \delta \\ \cos\alpha &= 1 \\ \cos\delta &= 1 \\ \alpha_r &\approx \beta + b\frac{r}{v} \end{aligned} \tag{6.4.4}$$

By substituting the previous equations in Eq. 6.4.4, we get

$$F_{yf} = C_f\left(\beta + \delta - a\frac{r}{v}\right) \tag{6.4.5}$$

$$F_{yr} = C_r\left(\beta + b\frac{r}{v}\right) \tag{6.4.6}$$

Noting that $\dot{\gamma} = v/R$, we obtain the following relationship among yaw angle, forward velocity, and sideslip angle:

$$\begin{aligned} \psi &= \beta + \gamma \\ r &= \dot{\beta} + \dot{\gamma} \end{aligned} \tag{6.4.7}$$

Substituting Eqs. 6.4.5 and 6.4.6 into Eqs. 6.4.1 and 6.4.2 gives

$$\begin{aligned} &C_r\left(\beta + b\frac{r}{v}\right) + C_f\left(\beta + \delta - a\frac{r}{v}\right) - F_w + mv\left(\dot{\beta} - r\right) + m\dot{v}\beta = 0 \\ &C_f a\left(\beta + \delta - a\frac{r}{v}\right) + M_w - I\dot{r} - C_r b\left(\beta + b\frac{r}{v}\right) = 0 \end{aligned} \tag{6.4.8}$$

The previous set of equations is nonlinear. By assuming the forward velocity to be constant (namely, $\dot{v} = 0$), we can rearrange these equations as

$$r\left(C_r\frac{b}{v} - C_f\frac{a}{v} - mv\right) + \dot{\beta}mv + \beta\left(C_f + C_r\right) + \delta C_f - F_w = 0 \tag{6.4.9}$$

$$\dot{r}I + r\left(C_f\frac{a^2}{v} + C_r\frac{b^2}{v}\right) + \beta\left(C_r b - C_f a\right) - \delta a C_r - M_w = 0 \tag{6.4.10}$$

If the forward velocity V, the cornering stiffness C_f and C_r, and parameters a, b, m, and I are known, with given steer angle δ, the sideslip angle β, the yaw or vehicle heading ψ, and the direction angle of velocity γ can be calculated. Generally, $C_f >> F_{xf}$. Thus, the longitudinal force will not affect the lateral motion. We have

$$\dot{\beta} = -\left(\frac{C_r + C_f}{mv}\right)\beta + \left(\frac{C_f b - C_r a}{mv^2} - 1\right)r + \frac{C_f}{mv}\delta + \frac{F_w}{mv} \tag{6.4.11}$$

$$\dot{r} = -\left(\frac{C_r b - C_f a}{I}\right)\beta - \left(\frac{C_r b^2 + C_f a^2}{I}\right)r + \frac{aC_f}{I}\delta + \frac{M_w}{I} \tag{6.4.12}$$

As a state equation, we have

$$P\dot{X} + QX = RU + W \tag{6.4.13}$$

where X is a state vector, W is a disturbance, and U is the input control vector.

$$
\begin{aligned}
X &= \begin{bmatrix} \beta \\ r \end{bmatrix} \\
U &= [\delta] \\
W &= \begin{bmatrix} F_w/v \\ M_w \end{bmatrix} \\
U &= [\delta] \\
P &= \begin{bmatrix} m & \\ & I \end{bmatrix} \\
Q &= \begin{bmatrix} \dfrac{C_r + C_f}{v} & \dfrac{-C_r b + C_f a}{v} + v \\ \left(\dfrac{-C_r b + C_f a}{v}\right)\beta & \dfrac{+C_r b^2 + C_f a^2}{v} \end{bmatrix} \\
R &= \begin{bmatrix} \dfrac{C_f}{v} \\ \dfrac{C_f a}{I} \end{bmatrix}
\end{aligned}
\tag{6.4.14}
$$

Problem 6.5

Implement the model developed in Problem 6.4 using MATLAB programming, where the model can be a function of states and control as input and state derivatives as output.

Solution to Problem 6.5

The model developed in Problem 6.4 can be expressed as

$$
\begin{aligned}
\dot{\beta} &= a_{11}\beta + a_{12}r + b_1\delta \\
\dot{r} &= a_{21}\beta + a_{22}r + b_2\delta
\end{aligned}
\tag{6.5.1}
$$

For convenience, we assume that the interested time span is T and that the time step is dt. We assume

$$
\begin{aligned}
t &= x \\
\beta &= y_1 \\
r &= y_2 \\
\delta &= f(t) \\
\beta(0) &= y_{10} \\
r(0) &= y_{20}
\end{aligned}
\tag{6.5.2}
$$

The MATLAB implementation is as follows:

```
tspan=[0: dt: T]
y0=[y10,y20];
[t,y]=ode23('func6_4,tspan,y0);
subplot(211)
plot(t,y(:,1))
xlabel('t')
ylabel('β')
subplot(212)
plot(t,y(:,2))
xlabel('t')
ylabel('r')

function f=func6_4(t,y)
f=zeros(2,1)
f(1)=a11*y(1)+a12*y(2)+b1*f(t);
f(2)=a21*y(1)+a22*y(2)+b2*f(t);
```

Problem 6.6

Assume that a car is driven on a road with a constant radius of 30 m at a velocity of 15 m/s, where L = 2.8 m. Determine the required steering angle.

Solution to Problem 6.6

Based on R = 30 m and V = 15 m/s, we have

$$r = V/R = 15/30 = 0.5 \text{ rad/s}$$

Consider K = 0.45. (Refer to E6.3 in the book *Road Vehicle Dynamics* [see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7].)

$$\delta = \frac{\left(1 + Kv^2\right)r}{\left(v/L\right)} = \frac{\left(1 + 0.45 \times 15^2\right) \times 0.5}{15/2.8} = 9.5 \text{ rad}$$

Now consider K = 0.265. (Again, refer to E6.3 in the book *Road Vehicle Dynamics* [see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7].)

$$\delta = \frac{\left(1 + Kv^2\right)r}{\left(v/L\right)} = \frac{\left(1 + 0.265 \times 15^2\right) \times 0.5}{15/2.8} = 5.7 \text{ rad}$$

Problem 6.7

Consider a car that has the following parameters:

Mass:	M = 2045 kg
Inertia:	I = 5428 kgm · m
Distance from the front axle to CG:	a = 1.49 m
Distance from the rear axle to CG:	a = 1.7 m
Front cornering stiffness:	C_f = 77.9 kN/rad
Rear cornering stiffness:	C_r = 76.5 kN/rad

Determine the understeer gradient K.

Solution to Problem 6.7

$$K = \frac{m\left(bC_r - aC_f\right)}{L^2 C_f C_r} = 0.9 \text{ deg/g}$$

Problem 6.8

For Problem 6.7, assume the following:

Velocity:	=	20 m/s
Lateral acceleration:	=	0.3 g
Lateral angular velocity:	=	8.4°/s
Radius:	=	136 m

Determine the stable response characteristic parameters ω_d and ξ.

Solution to Problem 6.8

Based on the parameters given in Problem 6.7, we have

$$a_{11} = -\frac{C_r + C_f}{mv}$$

$$a_{12} = -\left(\frac{C_r b - C_f a}{mv^2} - 1\right)$$

$$a_{21} = -\frac{C_r b - C_f a}{I}$$

$$a_{22} = -\frac{C_r b^2 + C_f a^2}{Iv}$$

$$b_1 = \frac{C_f + F_{xf}}{mv}$$

$$b_2 = \frac{a\left(C_f + F_{xf}\right)}{I}$$

The motion equation can be expressed as

$$\dot{\beta} = a_{11}\beta + a_{12}r + b_1\delta$$
$$\dot{r} = a_{21}\beta + a_{22}r + b_2\delta$$

Eliminating one variable β gives

$$\ddot{r} - \left(a_{11} + a_{22}\right)\dot{r} + \left(a_{11}a_{22} - a_{12}a_{21}\right)r = b_2\dot{\delta} + \left(a_{21}b_1 - a_{11}b_2\right)\delta$$

By assuming $\omega_n^2 = a_{11}a_{22} - a_{12}a_{21}$ and $2\xi\omega_n = -\left(a_{11} + a_{22}\right)$, we define ωn as the natural circular frequency of yaw velocity, and ξ as the damping factor of yaw velocity. Hence, we obtain

$$\omega_n = \sqrt{\frac{mv\left(aC_f - bC_r\right) + C_f C_r L^2/v}{mvI}} = \frac{L}{v}\sqrt{\frac{C_f C_r\left(1 + Kv^2\right)}{mI}}$$

$$\omega_d = \omega_n\sqrt{1 - \xi^2}$$

The damping factor or ratio can be derived as

$$\xi = \frac{-\left[m\left(a^2C_f + b^2C_r\right) + I\left(C_f + C_r\right)\right]}{2mIL\sqrt{\dfrac{C_fC_r\left(1+Kv^2\right)}{mI}}} = \frac{-\left[m\left(a^2C_f + b^2C_r\right) + I\left(C_f + C_r\right)\right]}{2L\sqrt{mIC_fC_r\left(1+Kv^2\right)}}$$

Finally, we obtain

$$\xi = 0.91$$
$$\omega_d = 1.63 \text{ rad/s}$$

Problem 6.9

Determine the function of the eigenvalue of the above system as a function of velocity (from 20 to 50 m/s).

Solution to Problem 6.9

Following the procedure in Problem 6.8, for a velocity of 20 m/s, we obtain

$$\xi = 0.91$$
$$\omega_d = 1.63 \text{ rad/s}$$

For a velocity of 50 m/s, we obtain

$$\xi = 0.67$$
$$\omega_d = 1.70 \text{ rad/s}$$

Problem 6.10

Calculate and plot the response of the car as described in Problem 6.7 under step steer input and under the condition of velocity v = 50 m/s.

Solution to Problem 6.10

The solved response of the car in Problem 6.7 under step steer input and under the condition of velocity v = 50 m/s is plotted in Figure 6.10.

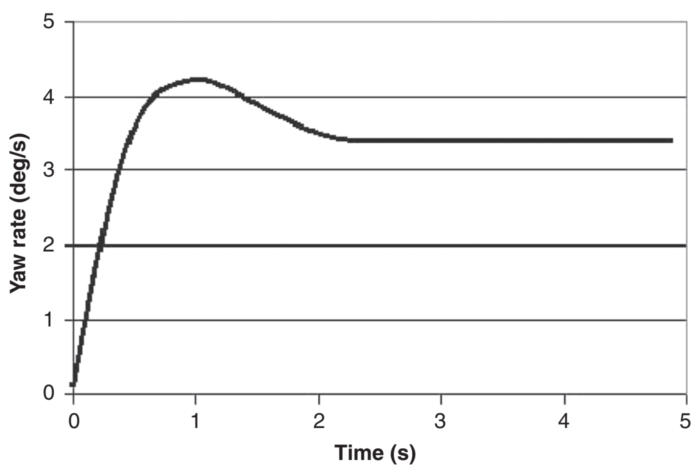

Figure 6.10 Plot of car in Problem 6.7 under step steer input where v = 50 m/s.

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Problem 6.11

Calculate and plot the frequency response of the car in Problem 6.7 under the conditions of velocity v = 20 m/s and v = 50 m/s.

Solution to Problem 6.11

The calculated frequency response of the car in Problem 6.7 under the conditions of velocity v = 20 m/s and v = 50 m/s is plotted in Figure 6.11.

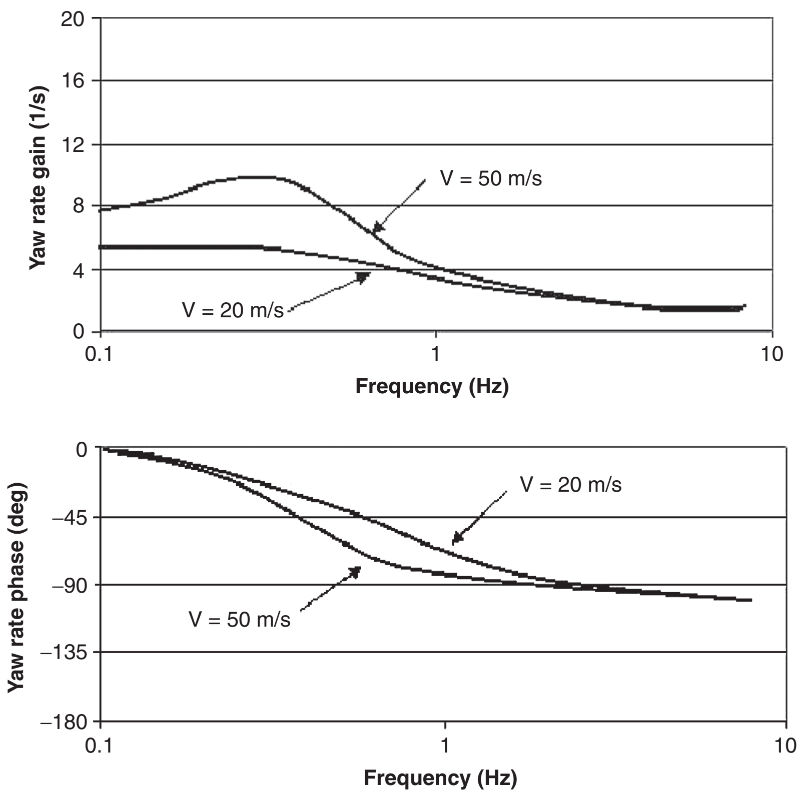

Figure 6.11 Frequency response of car in Problem 6 where v = 20 m/s and v = 50 m/s.

Problem 6.12

Consider a car that has the following parameters:

Mass:	M = 1008 kg
Inertia:	I = 1031 kgm · m
Distance from the front axle to CG:	a = 1.2 m
Distance from the rear axle to CG:	a = 1 m
Front cornering stiffness:	C_f = 117 kN/rad
Rear cornering stiffness:	C_r = 145 kN/rad

Determine the understeer gradient K.

Solution to Problem 6.12

$$K = \frac{m\left(bC_r - aC_f\right)}{L^2 C_f C_r} = 0.05 \text{ deg/g}$$

Problem 6.13

For Problem 6.12, assume the following:

Velocity:	=	20 m/s
Lateral acceleration:	=	0.3 g
Lateral angular velocity:	=	8.4 deg/s
Radius:	=	136 m

Determine the stable response characteristic parameters.

Solution to Problem 6.13

Based on the parameters given for Problem 6.12, we have

$$a_{11} = -\frac{C_r + C_f}{mv}$$

$$a_{12} = -\left(\frac{C_r b - C_f a}{mv^2} - 1\right)$$

$$a_{21} = -\frac{C_r b - C_f a}{I}$$

$$a_{22} = -\frac{C_r b^2 + C_f a^2}{Iv}$$

$$b_1 = \frac{C_f + F_{xf}}{mv}$$

$$b_2 = \frac{a\left(C_f + F_{xf}\right)}{I}$$

The motion equation can be expressed as

$$\dot{\beta} = a_{11}\beta + a_{12}r + b_1\delta$$

$$\dot{r} = a_{21}\beta + a_{22}r + b_2\delta$$

By eliminating one variable β, we obtain

$$\ddot{r} - \left(a_{11} + a_{22}\right)\dot{r} + \left(a_{11}a_{22} - a_{12}a_{21}\right)r = b_2\dot{\delta} + \left(a_{21}b_1 - a_{11}b_2\right)\delta$$

By assuming $\omega_n^2 = a_{11}a_{22} - a_{12}a_{21}$ and $2\xi\omega_n = -\left(a_{11} + a_{22}\right)$, we define ωn as the natural circular frequency of the yaw velocity, and we define ξ as the damping factor of the yaw velocity. Hence,

$$\omega_n = \sqrt{\frac{mv\left(aC_f - bC_r\right) + C_f C_r L^2/v}{mvI}}$$

$$= \frac{L}{v}\sqrt{\frac{C_f C_r\left(1 + Kv^2\right)}{mI}}$$

$$\omega_d = \omega_n\sqrt{1 - \xi^2}$$

The damping factor or ratio can be derived as

$$\xi = \frac{-\left[m\left(a^2C_f + b^2C_r\right) + I\left(C_f + C_r\right)\right]}{2mIL\sqrt{\dfrac{C_fC_r\left(1+Kv^2\right)}{mI}}}$$

$$= \frac{-\left[m\left(a^2C_f + b^2C_r\right) + I\left(C_f + C_r\right)\right]}{2L\sqrt{mIC_fC_r\left(1+Kv^2\right)}}$$

Finally, we obtain

$$\xi = 1$$
$$\omega_d = 0.88 \text{ rad/s}$$

Problem 6.14

Determine the function of the eigenvalue of the system in Problem 6.12 as a function of velocity (from where v = 20 m/s to v = 50 m/s).

Solution to Problem 6.14

Following the procedure used in Problem 6.13, for velocity v = 20 m/s, we obtain

$$\xi = 1$$
$$\omega_d = 0.88 \text{ rad/s}$$

For velocity v = 50 m/s, we obtain

$$\xi = 0.96$$
$$\omega_d = 1.63 \text{ rad/s}$$

Problem 6.15

Calculate and plot the response of the car discussed in Problem 6.12 under step steer input and under the condition of velocity v = 50 m/s.

Solution to Problem 6.15

The solved response of the car from Problem 6.12 under step steer input and under the condition of velocity v = 50 m/s is plotted in Figure 6.15.

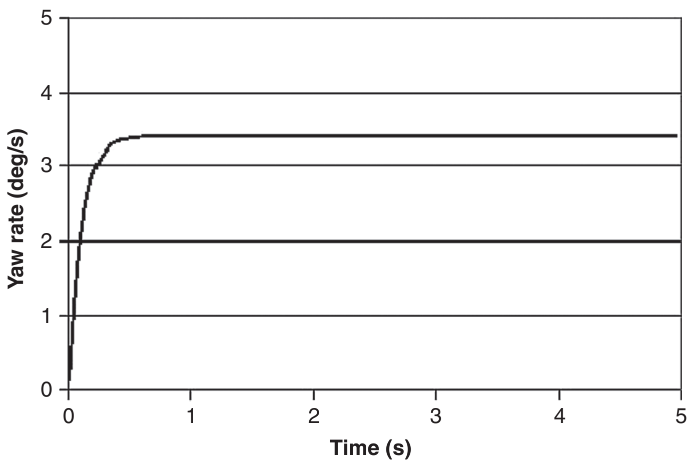

Figure 6.15 Car under step steer input and where velocity v = 50 m/s.

Problem 6.16

Calculate and plot the frequency response of the car from Problem 6.12 under the conditions of velocity v = 20 and v = 50 m/s.

Solution to Problem 6.16

The calculated frequency response of the car from Problem 6.12 under the conditions of velocity v = 20 and v = 50 m/s is plotted in Figure 6.16.

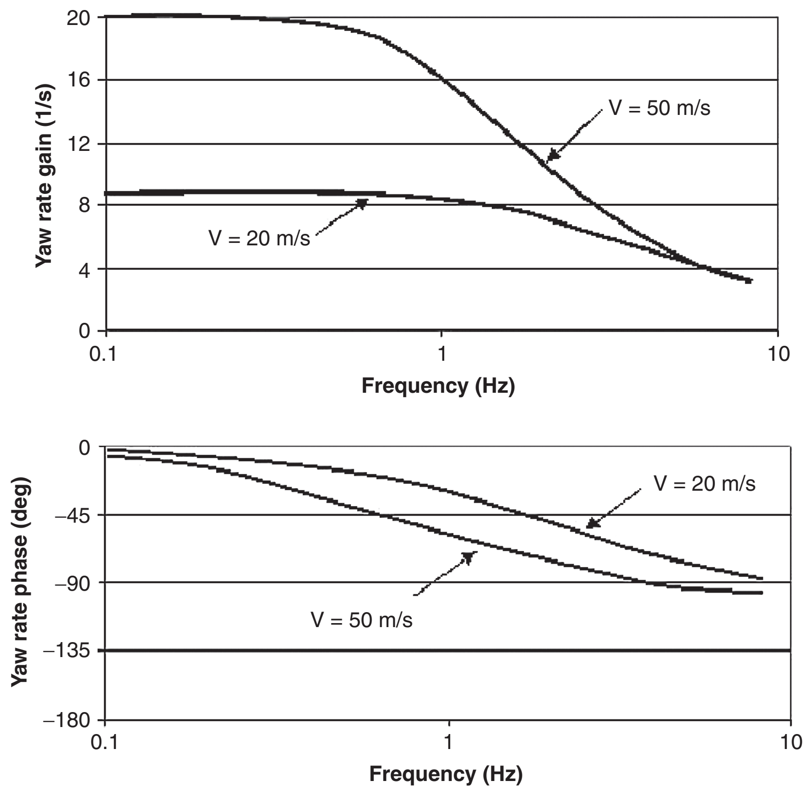

Figure 6.16 Calculated frequency response of the car from Problem 6.12, where velocity v = 20 m/s and v = 50 m/s.

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Chapter 7

Braking

Problem 7.1

A 18,683-N (4200-lb) vehicle traveling at 104.61 km/hr (65 mph) is stopped under a constant braking force. Find the braking force needed to bring the vehicle to a complete stop in 9 s, 7 s, and 5 s. Ignore drag, rolling resistance, and grade.

Solution to Problem 7.1

Stopping in 9 s:

The initial velocity is

$$V_i = \frac{104.61\ \text{km}}{\text{hr}} \times \frac{1000\ \text{m}}{\text{km}} \times \frac{\text{hr}}{3600\ \text{s}} = 29.06\ \text{m/s}$$

The acceleration is

$$a = \left(V_f - V_i\right)/t = \left(0 - 29.06\right)/9 = -3.23\ \text{m/s}^2\ \text{T}$$

The braking force is

$$F_b = Ma = -18638 * 3.23/9.81 = -6135\ \text{N}$$

Stopping in 7 s:

The acceleration is

$$a = \left(V_f - V_i\right)/t = \left(0 - 29.06\right)/7 = -4.15\ \text{m/s}^2$$

The braking force is

$$F_b = Ma = -18638 * 4.15/9.81 = -7885\ \text{N}$$

Stopping in 5 s:

The acceleration is

$$a = \left(V_f - V_i\right)/t = \left(0 - 29.06\right)/5 = -5.81 \text{ m/s}^2$$

The braking force is

$$F_b = Ma = -18638 * 5.81/9.81 = -11{,}042 \text{ N}$$

Problem 7.2

For Problem 7.1, assume that the vehicle has a frontage area of 22 ft^2 and a coefficient of drag of 0.35. Given that the drag force itself changes as the velocity decreases (Eq. 7.27), determine the braking force needed to bring the vehicle to a complete stop within 5 s.

Solution to Problem 7.2

Assume that the density of air is 1.226 kg/m^3 (0.002378 $slug/ft^3$).

The frontage area is

$$A_F = 22 \text{ ft}^2 = 22\frac{\left(0.3048 \text{ m}\right)^2}{\text{ft}^2} = 2.044 \text{ m}^2$$

The air resistance is

$$\begin{aligned} R_a &= \frac{\rho}{2} C_D A_F V_r^2 \\ &= \frac{1.226}{2} \times 0.35 \times 2.044 \times \left(V\right)^2 = 0.4385 \ V^2 \text{ lb} \end{aligned}$$

The total force is

$$F = -F_b - R_a = -F_b - 0.4385 \ V^2$$

The mass is

$$m = \frac{18638}{9.81} = 1900.0 \text{ kg}$$

The acceleration is

$$\begin{aligned} a = F/m &= \frac{-F_b - 0.4385 \ V^2}{1900} \\ &= -0.0005263 \ F_b - 0.0002308 \ V^2 \end{aligned}$$

We will solve this by dividing the time increments into steps. We will assume constant acceleration in the previous step. Thus, the velocity at any time t_i or V_i is

$$V_i = V_{i-1} + a_{i-1}\left(\Delta t\right)$$

The force F_b will be changed until the velocity reaches zero after 5 s. The following table was developed with the use of Excel. Note that the force needed to achieve a complete stop in 5 s is 10,913 N, which is slightly less than the 11,042 N needed when no air resistance is present to impede the car.

	F_b = 10913	Del t = 0.2
t	V =	a
0	29.06	–5.94
0.2	27.87	–5.92
0.4	26.69	–5.91
0.6	25.51	–5.89
0.8	24.33	–5.88
1	23.15	–5.87
1.2	21.98	–5.85
1.4	20.81	–5.84
1.6	19.64	–5.83
1.8	18.47	–5.82
2	17.31	–5.81
2.2	16.14	–5.80
2.4	`14.98	–5.80
2.6	13.83	–5.79
2.8	12.67	–5.78
3	11.51	–5.77
3.2	10.36	–5.77
3.4	9.20	–5.76
3.6	8.05	–5.76
3.8	6.90	–5.75
4	5.75	–5.75
4.2	4.60	–5.75
4.4	3.45	–5.75
4.6	2.30	–5.74
4.8	1.15	–5.74
5	0.00	–5.74

Problem 7.3

A 19,572.2-N (4400-lb) vehicle traveling at 104.61 km/hr (65 mph) is stopped under a constant braking force of 889.6 N (200 lb). For the following, find the time needed for travel before the vehicle reaches a complete stop:

a. When the vehicle is moving on a horizontal plane

b. When the vehicle is moving uphill on a grade of 6%

c. When the vehicle is going downhill on a grade of 6%

Solution to Problem 7.3

Assume no resistance.

The mass is

$$m = \frac{19,572.2}{9.81} = 1995 \text{ kg}$$

The speed is

$$V_i = \frac{104.61 \text{ km}}{\text{hr}} \times \frac{1000 \text{ m}}{\text{km}} \times \frac{\text{hr}}{3600\,\text{s}} = 29.06 \text{ m/s}$$

a. When the vehicle is moving on a horizontal plane,

The total force is 889.6 N.

The acceleration is

$$a = F/m = \frac{-889.6}{1995} = -0.446 \text{ m/s}^2$$

The time is

$$t = \frac{V_f - V_i}{a} = aF/m = \frac{-29.06}{-0.446} = 65.17 \text{ s}$$

b. When the vehicle is moving uphill on a grade of 6%,

The total force is

$$F = F_b - W\sin\theta = -889.6 - 19572.2\sin(6) = -2935.5\text{N}$$

The acceleration is

$$a = F/m = \frac{-2935.5}{1893.2} = -1.550 \text{ m/s}^2$$

The time is

$$t = \frac{V_f - V_i}{a} = \frac{-29.06}{-1.550} = 18.74 \text{ s}$$

c. When the vehicle is moving downhill on a grade of 6%,

The total force is

$$F = F_b - W\sin\theta = -889.6 + 19572.2\sin(6) = 1156.2\text{N}$$

The acceleration is

$$a = F/m = \frac{1156.2}{1893.2} = 0.610 \text{ m/s}^2$$

The acceleration is positive, which means that the speed of the car will continue to increase. The braking force will not be enough to stop the car.

Problem 7.4

Consider the brake system shown in Figure 7.4. The width of the shoe system is 31.75 mm (1.25 in.), and the coefficient of friction is 0.4. The vehicle specifications call for a braking force of 4448.2 N (1000 lb). Find the maximum pressure at the brake shoes allowed for achieving the desired braking force. Also find the total torque applied by the brake system, as well as the reaction force at the pivots of both shoes.

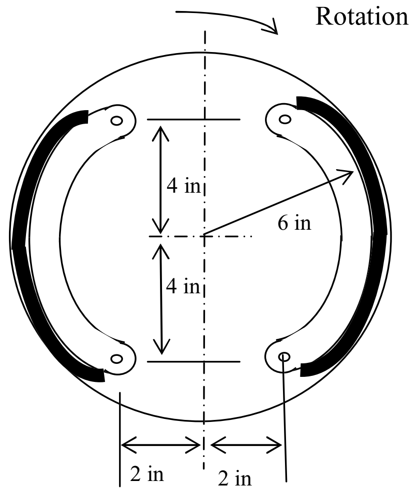

Figure 7.4 Brake for Problem 7.4.

Solution to Problem 7.4

First we will analyze the right-hand shoe. The distance (a_1) from the pivot to the center is

$$a_1 = \sqrt{(4)^2 + (2)^2} = 4.472 \text{ in.}$$

The angles are

$$\theta_2 = 180 - 2\tan^{-1}\left(\frac{2}{4}\right) = 180 - 2 \times 26.5 = 127°$$

Next, we need to find the moment from the friction forces

$$\begin{aligned} M_F &= \mu p_{max} br\left[\left(-r\cos\theta\right)_0^{\theta_2} - a_1\left(\frac{\sin^2\theta}{2}\right)_0^{\theta_2}\right] \\ &= \mu p_{max} br\left[r - r\cos\theta_2 - \frac{a_1}{2}\sin^2\theta_2\right] \\ &= 0.4 \times p_{max} \times 1.25 \times 6\left[6 - 6\cos 127° - \frac{4.472}{2}\sin^2 127°\right] \\ &= 24.554\ p_{max} \end{aligned}$$

The moment from the normal forces is

$$M_N = p_{max}\,bra_1\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\theta_2} = p_{max}\,bra_1\left[\frac{\theta_2}{2} - \frac{1}{4}\sin 2\theta_2\right]$$

$$= p_{max} \times 1.25 \times 6 \times 4.472\left[\frac{1}{2}\frac{\pi \times 127}{180} - \frac{1}{4}\sin\left(2 \times 127°\right)\right]$$

$$= 45.232\ p_{max}$$

The applied or actuation force is

$$F = \frac{M_N - M_f}{e}$$

$$= \frac{45.232 - 24.554}{8}p_{max}$$

$$= 2.585\ p_{max} = 1000\ \text{lb}$$

or

$$p_{max} = 386.9\ \text{psi}$$

Thus,

$$M_F = 24.554\ p_{max} = 9500\ \text{lb}\cdot\text{in.}$$

The moment from the normal forces is

$$M_N = 45.232\ p_{max} = 17{,}500\ \text{lb}\cdot\text{in.}$$

The reactions at the pivot of the right shoe are

$$R_x = p_{max}\,br\left(\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\ d\theta - \mu\int_{\theta_1}^{\theta_2}\sin^2\theta\ d\theta\right) - F_x$$

$$= 386.9 \times 1.25 \times 6\left(\frac{1}{2}\sin^2 127 - 0.4\left(\frac{\pi \times 127}{2 \times 180} - \frac{1}{4}\sin\left(2 \times 127\right)\right)\right) - 1000\sin 26.5$$

$$= 2901.8\left(0.319 - 0.4\left(1.34\right)\right) - 1000\sin 26.5$$

$$= -1086\ \text{lb}$$

$$R_y = p_{max}\,br\left(\mu\int_{\theta_1}^{\theta_2}\sin\theta\ \cos\theta\ d\theta + \int_{\theta_1}^{\theta_2}\sin^2\theta\ d\theta\right) - F_y$$

$$= 2901.8\left(0.4\left(0.319\right) + \left(1.34\right)\right) - 1000\cos 26.5$$

$$= 3364\ \text{lb}$$

The resultant force at the right pivot is

$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{\left(-1086\right)^2 + \left(3364\right)^2}$$

$$= 3535\ \text{lb}$$

Next, we will find the torque of the right-hand shoe (T_R):

$$T_R = \mu p_{max} br^2(\cos\theta_1 - \cos\theta_2)$$
$$= 0.4 \times 386.9 \times 1.25 \times (6)^2(\cos(0) - \cos(127))$$
$$= 11{,}155 \text{ lb}\cdot\text{in.}$$

For the left shoe,

$$M_F = \mu p_{max} br\left[(-r\cos\theta)_0^{\theta_2} - a_1\left(\frac{\sin^2\theta}{2}\right)_0^{\theta_2}\right]$$
$$= \mu p_{max} br\left[r - r\cos\theta_2 - \frac{a_1}{2}\sin^2\theta_2\right]$$
$$= 0.4 \times p_{max} \times 1.25 \times 6\left[6 - 6\cos 127° - \frac{4.472}{2}\sin^2 127°\right]$$
$$= 24.55\, p_{max} \text{ lb}\cdot\text{in.}$$

The moment from the normal forces is

$$M_N = p_{max} bra_1\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\theta_2}$$
$$= p_{max} bra_1\left[\frac{\theta_2}{2} - \frac{1}{4}\sin 2\theta_2\right]$$
$$= p_{max} \times 1.25 \times 6 \times 4.472\left[\frac{1}{2}\frac{\pi \times 127}{180} - \frac{1}{4}\sin(2 \times 127°)\right]$$
$$= 45.22\, p_{max} \text{ lb}\cdot\text{in.}$$

The applied or actuation force is

$$F = \frac{M_N + M_f}{e} = \frac{(24.55 + 45.22)p_{max}}{8} = 1000$$

or

$$p_{max} = 114.66 \text{ psi}$$

Thus, the torque on the left shoe is

$$T_L = \mu p_{max} br^2(\cos\theta_1 - \cos\theta_2)$$
$$= 0.4 \times 114.66 \times 1.25 \times (6)^2(\cos(0) - \cos(127))$$
$$= 3306 \text{ lb}\cdot\text{in.}$$

The total torque on the shoes is

$$T = T_L + T_R = 14{,}461 \text{ lb}\cdot\text{in.}$$

Finally, we need to find the reaction forces on the pivot of the left shoe:

$$R_x = p_{max} br \left(\int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta + \mu \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta \right) - F_x$$

$$= 114.66 \times 1.25 \times 6 \left(\frac{1}{2}\sin^2 127 + 0.4\left(\frac{\pi \times 127}{2 \times 180} - \frac{1}{4}\sin(2 \times 127) \right) \right)$$

$$-1000 \sin 26.5$$

$$= 860(0.319 - 0.4(1.34)) - 1000 \sin 26.5$$

$$= -832.8 \text{ lb}$$

$$R_y = p_{max} br \left(\mu \int_{\theta_1}^{\theta_2} \sin\theta \cos\theta \, d\theta - \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta \right) - F_y$$

$$= 114.66 \times 1.25 \times 6 \left(-0.4\left(\frac{1}{2}\sin^2 127 \right) + \left(\frac{\pi \times 127}{2 \times 180} - \frac{1}{4}\sin(2 \times 127) \right) \right)$$

$$-1000 \cos 26.5$$

$$= 860(0.4 \times 0.319 + (1.34)) - 1000 \cos 26.5$$

$$= 2157 \text{ lb}$$

The resultant force at the right pivot is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-832.8)^2 + (2157)^2} = 2312 \text{ lb}$$

Problem 7.5

In Example E 7.3, the width of the shoe system is 30 mm and the coefficient of friction is 0.3. The friction material used permits a maximum pressure value of 1000 kPa. Find the actuation force, the total torque applied by the brake system, and the reaction force at the pivots of both shoes for the angles ψ = 20, 25, 30, 35, and 40°. The distance between the center of the brake and the pivots is constant at 110 mm.

Solution to Problem 7.5

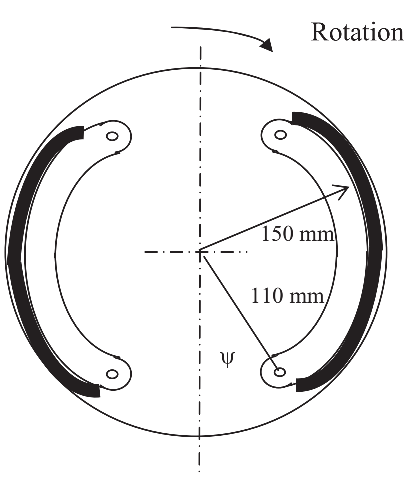

Figure 7.5 Brake for Problem 7.5.

Example		epsii	20	25	30	35	40 degrees
x	0.05 m	x	0.03762	0.04649	0.05500	0.06309	0.07071 m
y	0.1 m	y	0.10337	0.09969	0.09526	0.09011	0.08426 m
mue	0.3	mue	0.30	0.30	0.30	0.30	0.30
b	0.03 m	b	0.03	0.03	0.03	0.03	0.03 m
Pmax	1000000 N/m^2 (Pa)	Pmax	1000000	1000000	1000000	1000000	1000000 N/m^2 (Pa)
r	0.15 m	r	0.15	0.15	0.15	0.15	0.15 m
a1	0.111803 mm	a1	0.11	0.11	0.11	0.11	0.11 mm
theta2	2.214295 rad	theta2	2.4435	2.2689	2.0944	1.9199	1.7453 rad
e	0.2	e	0.2067	0.1994	0.1905	0.1802	0.1685
Right Shoe		**Right Shoe**					
MF	275.7003 Nm	MF	326.95	289.09	248.06	206.19	165.65 Nm
MN	677.7703 Nm	MN	726.63	683.43	625.53	554.71	474.29 Nm
F	2010.35 N	F	1933.32	1977.74	1981.21	1933.91	1831.37 N
Rx	−1277.7 N	Rx	−1713.29	−1379.37	−1009.10	−635.29	−288.55 N
Ry	4696.052 N	Ry	5067.86	4816.66	4477.14	4054.70	3563.49 N
R	4866.766 N	R	5349.63	5010.28	4589.45	4104.16	3575.15 N
TR	323.9996 Nm	TR	357.62	332.66	303.75	271.76	237.66 Nm
Left Shoe		**Left Shoe**					
MF	0.000276 *Pmax	MF	0.000327	0.000289	0.000248	0.000206	0.000166 *Pmax
MN	0.000678 *Pmax	MN	0.000727	0.000683	0.000626	0.000555	0.000474 *Pmax
Pmax	421691 Pa	Pmax	379358.1	405478.7	432089.0	458028.4	482292.5 Pa
TL	136.6277 Nm	TL	135.7	134.9	131.2	124.5	114.6 Nm
Total Torque		**Total Torque**					
T	460.6273 Nm	T	493.3	467.6	435.0	396.2	352.3 Nm

Problem 7.6

A vehicle that has a mass of 2200 kg and is traveling at 120 km/hr is to reach a complete stop in 10 s. The braking system has a mass of 10 kg. Let $C = 500\ \mathrm{J}/(\mathrm{kg}\cdot \mathrm{C}^\circ)$. Determine the energy and the average power. Also, estimate the temperature rise of the brakes.

Solution to Problem 7.6

$$V_i = \frac{120\ \text{km}}{\text{hr}} \times \frac{1000\ \text{m}}{\text{km}} \times \frac{\text{hr}}{3600\ \text{s}} = 33.33\ \text{m/s}$$

$$\text{K.E.} = \frac{1}{2}M\left(V_i^2 - V_f^2\right) = \frac{1}{2} \times 2200\frac{\text{N}\cdot\text{s}^2}{\text{m}} \times \left(33.33\frac{\text{m}}{\text{s}}\right)^2 = 1{,}222{,}222\ \text{N}\cdot\text{m}$$

$$\text{Power} = \frac{1{,}222{,}222\ \text{N}\cdot\text{m}}{10\ \text{s}} = 122\ \text{kJ}$$

The temperature rise at the brake can be estimated using

$$\Delta\text{Temp} = \frac{122{,}222}{500 \times 10} = 122\ \text{C}°$$

Problem 7.7

Consider a light truck that weighs 17,792.89 N (4000 lb) under a certain load. The wheelbase of the vehicle is at 3.048 m (10 ft), and the center of gravity is at 1.2192 m (4 ft), from the front axle, and at 0.6604 m (26 in.) above the ground. Determine the load on each axle if the brakes are designed for all-wheel lock-up under these conditions with a coefficient of friction of 0.8.

Solution to Problem 7.7

The following are given in the problem:

$$\begin{aligned} W &= 4000 \text{ lb} \\ c_1 + c_2 &= 120 \text{ in.} \\ h &= 26 \text{ in.} \\ c_1 &= 48 \text{ in.} \end{aligned}$$

First, we will find the mass and c_2

$$\begin{aligned} M &= 3200/32.2 = 124.2 \text{ slugs} \\ c_2 &= 120 - 48 = 72 \text{ in.} \end{aligned}$$

Apply Eq. 7.25 to find the loading under no acceleration (i.e., a = 0).

$$\begin{aligned} F_{z1} &= Mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2} Ma \\ &= 4000\left(\frac{72}{120}\right) - \frac{26}{120} \times 124.2 \times 0 = 2400 \text{ lb} \end{aligned}$$

$$\begin{aligned} F_{z2} &= Mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2} Ma \\ &= 4000\left(\frac{48}{120}\right) + \frac{26}{120} \times 124.2 \times 0 = 1600 \text{ lb} \end{aligned}$$

For braking forces,

$$F_{x1} = \mu F_{z1} = 0.8 \times 2400 = 1920 \text{ lb}$$
$$F_{x2} = \mu F_{z2} = 0.8 \times 1600 = 1280 \text{ lb}$$

The total load on axle 1 is

$$F_1 = \sqrt{F_{x1}^2 + F_{z1}^2} = \sqrt{(1920)^2 + (2400)^2} = 3073.5 \text{ lb}$$

The total load on axle 2 is

$$F_2 = \sqrt{F_{x2}^2 + F_{z2}^2} = \sqrt{(1280)^2 + (1600)^2} = 2049.0 \text{ lb}$$

The total braking force is

$$F_x = 1920 + 1280 = 3200 \text{ lb}$$

Problem 7.8

Consider a midsize sedan that weighs 1600 kg under a certain load and is moving uphill at a 5° slope at 88 km/hr. The front area of the vehicle is 2.0 m^2, and the coefficient of drag is 0.4. The aerodynamic resistance is acting on the center of gravity (CG). The wheelbase of the vehicle is at 2.8 m, and the center of gravity is at 1.2 m, from the front axle, and at 0.5 m above the ground. Determine the load on each axle if the brake is under no acceleration, decelerating at 5 m/s^2. The density of air is 1.225 kg/m^3.

Solution to Problem 7.8

The velocity is

$$V_i = \frac{88 \text{ km}}{\text{hr}} \times \frac{1000 \text{ m}}{\text{km}} \times \frac{\text{hr}}{3600 \text{ s}} = 24.44 \text{ m/s}$$

First, the aerodynamic resistance is calculated as

$$R_a = \frac{\rho}{2} C_D A_F V_r^2$$
$$= \frac{1.225}{2} \times 0.4 \times 2.0 \times (24.44)^2 = 292.8 \text{ N}$$

The forces on the axles (when going uphill and with no acceleration) are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2\cos\theta - h_aR_a - hMa - hMg\sin\theta\right)$$

$$= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.6 \times \cos(5) - 0.5 \times 2928 - 0.5 \times 1600 \times (0) - 0.5 \times 1600 \times 9.81 \times \sin(5)\right)$$

$$= \frac{1}{2.8}\left(25{,}018 - 1464 - (0) - 684\right)$$

$$= 8168 \text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1\cos\theta + h_aR_a + hMa + hMg\sin\theta\right)$$

$$= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.2 \times \cos(5) + 0.5 \times 2928 + 0.5 \times 1600 \times (0) + 0.5 \times 1600 \times 9.81 \times \sin(5)\right)$$

$$= \frac{1}{2.8}\left(18{,}764 + 1464 + (0) + 684\right)$$

$$= 7468 \text{ N}$$

The forces on the axles (when going uphill and with deceleration at 5 m/s^2) are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2\cos\theta - h_aR_a - hMa - hMg\sin\theta\right)$$

$$= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.6 \times \cos(5) - 0.5 \times 2928 - 0.5 \times 1600 \times (-5) - 0.5 \times 1600 \times 9.81 \times \sin(5)\right)$$

$$= \frac{1}{2.8}\left(25{,}018 - 1464 + 4000 - 684\right)$$

$$= 9596 \text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1\cos\theta + h_aR_a + hMa + hMg\sin\theta\right)$$

$$= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.2 \times \cos(5) + 0.5 \times 2928 + 0.5 \times 1600 \times (-5) + 0.5 \times 1600 \times 9.81 \times \sin(5)\right)$$

$$= \frac{1}{2.8}\left(18{,}764 + 1464 - 4000 + 684\right)$$

$$= 6040 \text{ N}$$

Problem 7.9

For the situation problem described in Problem 7.8, what would be the loads on each axle under no acceleration and at 5 m/s^2 acceleration, if the vehicle were traveling down a hill at a 5° slope?

Solution to Problem 7.9

The forces on the axles (when going downhill and with no acceleration) are

$$
\begin{aligned}
F_{z1} &= \frac{1}{c_1 + c_2}\left(Mgc_2 \cos\theta - h_a R_a - hMa - hMg\sin\theta\right) \\
&= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.6 \times \cos(-5) - 0.5 \times 2928 - 0.5 \times 1600 \times (0) - 0.5 \times 1600 \times 9.81 \times \sin(-5)\right) \\
&= \frac{1}{2.8}\left(25{,}018 - 1464 - (0) + 684\right) \\
&= 8656 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2} &= \frac{1}{c_1 + c_2}\left(Mgc_1 \cos\theta + h_a R_a + hMa + hMg\sin\theta\right) \\
&= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.2 \times \cos(-5) + 0.5 \times 2928 + 0.5 \times 1600 \times (0) + 0.5 \times 1600 \times 9.81 \times \sin(-51)\right) \\
&= \frac{1}{2.8}\left(18{,}764 + 1464 + (0) - 684\right) \\
&= 6980 \text{ N}
\end{aligned}
$$

The forces on the axles (when going downhill and with acceleration at 5 m/s^2) are

$$
\begin{aligned}
F_{z1} &= \frac{1}{c_1 + c_2}\left(Mgc_2 \cos\theta - h_a R_a - hMa - hMg\sin\theta\right) \\
&= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.6 \times \cos(-5) - 0.5 \times 2928 - 0.5 \times 1600 \times (5) - 0.5 \times 1600 \times 9.81 \times \sin(-5)\right) \\
&= \frac{1}{2.8}\left(25018 - 1464 - 4000 + 684\right) \\
&= 7228 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2} &= \frac{1}{c_1 + c_2}\left(Mgc_1 \cos\theta + h_a R_a + hMa + hMg\sin\theta\right) \\
&= \frac{1}{2.8}\left(1600 \times 9.81 \times 1.2 \times \cos(-5) + 0.5 \times 2928 + 0.5 \times 1600 \times (+5) + 0.5 \times 1600 \times 9.81 \times \sin(-5)\right) \\
&= \frac{1}{2.8}\left(18{,}764 + 1464 + 4000 - 684\right) \\
&= 8409 \text{ N}
\end{aligned}
$$

Problem 7.10

A 13,344.7-N (3000-lb) compact passenger car has a 2.5908-m (102-in.) wheelbase and a CG at 1.016 m (40 in.) behind the front axle and 0.4572 m (18 in.) above ground. The car is moving on a horizontal plane. Determine which tires will lock up first if the car is moving on a surface with a coefficient of friction of 0.9, if the front brake force distribution factors are 0.4, 0.6, and 0.8. Ignore drag and rolling resistance.

Solution to Problem 7.10

$$M = 4800\text{ lb} = 93.17\text{ slugs}$$

$$c_1 + c_2 = 102\text{ in.}$$

$$c_1 = 40\text{ in.}$$

$$c_2 = 62\text{ in.}$$

$$\mu = 0.9$$

$$h = 18$$

$$\theta = 0$$

$$R_a = 0$$

$$R_r = 0$$

For the first distribution factor of $K_f = 0.4$,

Acceleration under front lock-up is

$$a_F = -\frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= -\frac{0.9\left(\frac{62}{102}\right)\left(1 + \frac{0.6}{0.4}\right)}{\left(-\frac{0.9 \times 18}{102}\left(1 + \frac{0.6}{0.4}\right) + 1\right)} g$$

$$= -\frac{1.368}{0.6029} g$$

$$= -2.268g$$

Acceleration under rear lock-up is

$$a_R = -\frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g$$

$$= -\frac{0.9\left(\frac{40}{102}\right)\left(1 + \frac{0.4}{0.6}\right)}{\left(\frac{0.9 \times 18}{102}\left(1 + \frac{0.4}{0.6}\right) + 1\right)} g$$

$$= -\frac{0.5882}{1.265} g$$

$$= -0.4650g$$

Because $|a_F| > |a_R|$, the rear tires will lock up first.

For the first distribution factor of $K_f = 0.6$,

Acceleration under front lock-up is

$$a_F = -\frac{0.9\left(\frac{62}{102}\right)\left(1+\frac{0.4}{0.6}\right)}{\left(-\frac{0.9\times 18}{102}\left(1+\frac{0.4}{0.6}\right)+1\right)}g$$
$$= -\frac{0.9118}{1.265}g$$
$$= -0.7210g$$

Acceleration under rear lock-up is

$$a_R = -\frac{0.9\left(\frac{40}{102}\right)\left(1+\frac{0.6}{0.4}\right)}{\left(\frac{0.9\times 18}{102}\left(1+\frac{0.6}{0.4}\right)+1\right)}g$$
$$= -\frac{0.8824}{1.3971}g$$
$$= -0.6316g$$

Because $|a_F| > |a_R|$, the rear tires will lock up first.

For the first distribution factor of $K_f = 0.8$,

Acceleration under front lock-up is

$$a_F = -\frac{0.9\left(\frac{62}{102}\right)\left(1+\frac{0.2}{0.8}\right)}{\left(-\frac{0.9\times 18}{102}\left(1+\frac{0.2}{0.8}\right)+1\right)}g$$
$$= -\frac{0.6838}{1.1985}g$$
$$= -0.5705g$$

Acceleration under rear lock-up is

$$a_R = -\frac{0.9\left(\frac{40}{102}\right)\left(1+\frac{0.8}{0.2}\right)}{\left(\frac{0.9\times 18}{102}\left(1+\frac{0.8}{0.2}\right)+1\right)}g$$
$$= -\frac{1.7647}{1.7941}g$$
$$= -0.9836g$$

Because $|a_F| < |a_R|$, the front tires will lock up first.

Problem 7.11

Consider a vehicle with a 1500-kg mass when lightly loaded (driver only), with a CG at 1.0 m behind the front axle and 0.45 m above the road. The mass is 2200 kg when loaded with a CG at 1.25 m behind the front axle and 0.5 m above the road. The wheelbase is 2.7 m. The vehicle is to achieve maximum possible braking force and front lock-up under the following severe conditions:

Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$

Case 2: Vehicle is loaded, on a slippery surface of $\mu = 0.2$

Case 3: Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$

Case 4: Vehicle is lightly loaded, on a slippery road of $\mu = 0.2$

Determine the value of the front brake force distribution factor K_f that is to be recommended, given that the brake system your company installs has a constant value (no feedback control system). Determine the axle loads and the maximum braking force.

Solution to Problem 7.11

For a lightly loaded vehicle,

$M = 1500$ kg

$c_1 + c_2 = 2.7$ m

$c_1 = 1.0$ m

$c_2 = 1.7$ m

$h = 0.45$ m

For a loaded vehicle,

$M = 2200$ kg

$c_1 + c_2 = 2.7$ m

$c_1 = 1.25$ m

$c_2 = 1.45$ m

$h = 0.45$ m

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 + \mu h)}{(c_1 + c_2)} = \frac{1.45 + 0.9 \times 0.5}{2.7} = 0.7037$$

For Case 2: Vehicle is loaded, on a slippery surface of $\mu = 0.2$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 + \mu h)}{(c_1 + c_2)} = \frac{1.45 + 0.2 \times 0.5}{2.7} = 0.5741$$

For Case 3: Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 + \mu h)}{(c_1 + c_2)} = \frac{1.7 + 0.9 \times 0.45}{2.7} = 0.7796$$

For Case 4: Vehicle is lightly loaded, on a slippery surface of $\mu = 0.2$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 + \mu h)}{(c_1 + c_2)} = \frac{1.7 + 0.2 \times 0.45}{2.7} = 0.6630$$

To ensure front lock-up, a distribution factor of 0.8 is selected.

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$,

Acceleration under front lock-up is

$$a_F = -\frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= -\frac{0.9\left(\frac{1.45}{2.7}\right)\left(1 + \frac{0.2}{0.8}\right)}{\left(-\frac{0.9 \times 0.5}{2.7}\left(1 + \frac{0.2}{0.8}\right) + 1\right)} g$$

$$= -\frac{0.6042}{0.7917} g$$

$$= -0.7632g$$

Acceleration under rear lock-up is

$$a_R = -\frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g$$

$$= -\frac{0.9\left(\frac{1.25}{2.7}\right)\left(1 + \frac{0.8}{0.2}\right)}{\left(\frac{0.9 \times 0.5}{2.7}\left(1 + \frac{0.8}{0.2}\right) + 1\right)} g$$

$$= -\frac{2.083}{1.838} g$$

$$= -1.1364g$$

Because $|a_F| < |a_R|$, a front lock-up is confirmed to occur first. The vertical load on each axis is

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{1}{2.7}\left(21582 \times 1.45 - 0.5 \times 21582 \times \left(-0.7632\right)\right)$$

$$= 14,640 \text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{1}{2.7}\left(21582 \times 1.25 + 0.5 \times 21582 \times \left(-0.7632\right)\right)$$

$$= 6942 \text{ N}$$

The braking force on each axle is

$$F_{x1} = -\mu F_{z1} = -0.9 \times 14,640 = -13,176 \text{ N}$$

$$F_{x2} = \frac{1 - K_f}{K_f} F_{x1} = -\frac{0.2}{0.8} \times 13,176 = -3294 \text{ N}$$

The total braking force is

$$F_x = F_{x1} + F_{x2} = -13,176 - 3294 = -16,470 \text{ N}$$

Similar calculations are performed for all remaining cases. The following table is generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	1.25	1.45	0.5	0.9	21582	0.8	−0.763	**14640**	**6942**	**−13176**	**−3294**	**−16470**	**−1.1364**
2	1.25	1.45	0.5	0.2	21582	0.8	−0.141	**12153**	**9429**	**−2431**	**−608**	**−3038**	**−0.3906**
3	1	1.7	0.45	0.9	14715	0.8	−0.872	**11403**	**3312**	**−10263**	**−2566**	**−12828**	**−0.9524**
4	1	1.7	0.45	0.2	14715	0.8	−0.164	**9668**	**5047**	**−1934**	**−483**	**−2417**	**−0.3175**

Problem 7.12

The vehicle described in Problem 7.11 has an anti-lock braking system (ABS) that allows control of each wheel force distribution. Determine the maximum braking force achieved under the four conditions described in Problem 7.11.

Solution to Problem 7.12

With ABS, each axle can reach lock-up condition. This means that the vehicle can achieve its maximum braking force.

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$,

$$F_x = -\mu F_z = 0.9 \times 2200 \times 9.81 = -19{,}423 \text{ N}$$

For Case 2: Vehicle is loaded, on a slippery surface of $\mu = 0.2$,

$$F_x = -\mu F_z = 0.2 \times 2200 \times 9.81 = -4316 \text{ N}$$

For Case 3: Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$,

$$F_x = -\mu F_z = 0.9 \times 1500 \times 9.81 = -13{,}243 \text{ N}$$

For Case 4: Vehicle is lightly loaded, on a slippery surface of $\mu = 0.2$,

$$F_x = -\mu F_z = 0.2 \times 1500 \times 9.81 = -2943 \text{ N}$$

Problem 7.13

For the vehicle described in Problem 7.11, determine the maximum braking force that can be achieved if the vehicle is moving on a split-mue surface with the right wheels on a 0.2 coefficient of friction surface and with the left wheels on a 0.8 surface.

Solution to Problem 7.13

Assume total symmetry for the right and left wheels. We will use the two loading conditions, with a load distribution factor of 0.8.

Note that for a split-mue condition, under front lock-up,

$$F_{x1} = \mu_L \frac{F_{z1}}{2} + \mu_R \frac{F_{z1}}{2} = \left(\frac{\mu_L + \mu_R}{2}\right) F_{z1}$$

Thus, the effective friction is

$$\mu = \left(\frac{\mu_L + \mu_R}{2}\right) = 0.5$$

For Case 1: Vehicle is loaded,

Acceleration under front lock-up is

$$a_F = -\frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= -\frac{0.5\left(\frac{1.45}{2.7}\right)\left(1 + \frac{0.2}{0.8}\right)}{\left(-\frac{0.5 \times 0.5}{2.7}\left(1 + \frac{0.2}{0.8}\right) + 1\right)} g$$

$$= -0.3796g$$

Acceleration under rear lock-up (for confirmation of front lock-up) is

$$a_R = -\frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}g$$

$$= -\frac{0.5\left(\frac{1.25}{2.7}\right)\left(1+\frac{0.8}{0.2}\right)}{\left(\frac{0.5\times 0.5}{2.7}\left(1+\frac{0.8}{0.2}\right)+1\right)}g$$

$$= -0.7911g$$

$$F_{z1} = \frac{1}{c_1+c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{1}{2.7}\left(21,582\times 1.45 - 0.5\times 21,582\times\left(-0.3796\right)\right)$$

$$= 13,107 \text{ N}$$

$$F_{z2} = \frac{1}{c_1+c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{1}{2.7}\left(21,582\times 1.25 + 0.5\times 21,582\times\left(-0.3796\right)\right)$$

$$= 8475 \text{ N}$$

The maximum braking force is

$$F_{x1} = -\mu F_{z1} = -0.5\times 13,107 = -6554 \text{ N}$$

$$F_{x2} = \frac{1-K_f}{K_f}F_{x1} = \frac{0.2}{0.8}\times(-6554) = -1638 \text{ N}$$

$$F_x = F_{x1} + F_{x2} = -6554 - 1638 = -8192 \text{ N}$$

For Case 2: Vehicle is lightly loaded,

Acceleration under front lock-up is

$$a_F = -\frac{\mu\left(\frac{c_2}{c_1+c_2}\right)\left(1+\frac{1-K_f}{K_f}\right)}{\left(-\frac{\mu h}{c_1+c_2}\left(1+\frac{1-K_f}{K_f}\right)+1\right)}g$$

$$= -\frac{0.5\left(\frac{1.7}{2.7}\right)\left(1+\frac{0.2}{0.8}\right)}{\left(-\frac{0.5\times 0.45}{2.7}\left(1+\frac{0.2}{0.8}\right)+1\right)}g$$

$$= -0.4393g$$

Acceleration under rear lock-up (for confirmation of front lock-up) is

$$a_R = -\frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}g$$

$$= -\frac{0.5\left(\frac{1.00}{2.7}\right)\left(1+\frac{0.8}{0.2}\right)}{\left(\frac{0.5\times 0.45}{2.7}\left(1+\frac{0.8}{0.2}\right)+1\right)}g$$

$$= -0.6536g$$

$$F_{z1} = \frac{1}{c_1+c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{1}{2.7}\left(14{,}715\times 1.7 - 0.45\times 14{,}715\times\left(-0.4393\right)\right)$$

$$= 10{,}342 \text{ N}$$

$$F_{z2} = \frac{1}{c_1+c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{1}{2.7}\left(14{,}715\times 1.0 + 0.5\times 14{,}715\times\left(-0.4393\right)\right)$$

$$= 4373 \text{ N}$$

The maximum braking force is

$$F_{x1} = -\mu F_{z1} = -0.5\times 10342 = -5171 \text{ N}$$

$$F_{x2} = \frac{1-K_f}{K_f}F_{x1} = \frac{0.2}{0.8}\times(-5171) = -1293 \text{ N}$$

$$F_x = F_{x1} + F_{x2} = -5171 - 1293 = -6464 \text{ N}$$

Problem 7.14

For Problem 7.13, an ABS is used in the braking action. Control of the braking force on each wheel is possible under this system. Determine the maximum braking force that can be achieved.

Solution to Problem 7.14

For Case 1: Vehicle is loaded,

$$F_x = -\mu F_z = 0.5\times 2200\times 9.81 = -10{,}791 \text{ N}$$

For Case 2: Vehicle is lightly loaded,

$$F_x = -\mu F_z = 0.5\times 1500\times 9.81 = -7358 \text{ N}$$

Problem 7.15

Perform the necessary manipulation of Eqs. 7.3 and 7.4 to establish Eq. 7.5.

Solution to Problem 7.15

$$\sum M_{Pivot} = eF + e_2\mu N_B - e_1 N_B = 0 \tag{7.3}$$

$$F_B = \mu N_B \tag{7.4}$$

Equation 7.4 can be rewritten as

$$N_B = \frac{F_B}{\mu}$$

Substituting this into Eq. 7.3 gives

$$eF + e_2 F_B - e_1 \frac{F_B}{\mu} = 0$$

Multiplying the above equation by μ/F gives

$$\mu e + \mu e_2 \frac{F_B}{F} - e_1 \frac{F_B}{F} = 0$$

or

$$\frac{F_B}{F}\left(\mu e_2 - e_1\right) = -\mu e$$

Multiplying by -1 and rearranging the above equation yields

$$\frac{F_B}{F} = \frac{\mu e}{e_1 - \mu e_2} \tag{7.5}$$

Problem 7.16

Consider a vehicle with a 2000-kg mass when lightly loaded (driver only), with a CG at 1.2 m behind the front axle and 0.5 m above the road. The wheelbase is 3.0 m. The vehicle is to achieve maximum possible braking force and front lock-up under the following severe conditions:

a. The front wheels are on a dry surface of $\mu = 0.9$, and the rear wheels are on a wet road of $\mu = 0.2$.

b. The rear wheels are on a dry surface of $\mu = 0.9$, and the front wheels are on a wet road of $\mu = 0.2$.

Determine the value of the front brake force distribution factor K_f that you recommend, given that the brake system your company installs has a constant value (no feedback control system). Determine the axle loads and the maximum braking force.

Solution to Problem 7.16

For a lightly loaded vehicle,

$M = 2000$ kg

$c_1 + c_2 = 3.0$ m

$c_1 = 1.2$ m

$c_2 = 1.8$ m

$h = 0.5$ m

Calculate K_{fmax} for this.

The maximum braking force occurs when both F_{x1} and F_{x2} approach their lock-up (or maximum values) simultaneously. This can be achieved only if we set $F_{x1} = -\mu_f F_{z1}$ and $F_{x2} = -\mu_r F_{z2}$. The total friction force is

$$F_x = F_{x1} + F_{x2} = -\mu_f F_{z1} - \mu_r F_{z2}$$

We will ignore aerodynamic forces, rolling resistance, and grade.

Recall that

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2 - hMa\right)$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1 + hMa\right)$$

The maximum deceleration is

$$a = \frac{F_{x1} + F_{x2}}{M} = \frac{-\mu_f F_{z1} - \mu_r F_{z2}}{M}$$

$$Ma = -\mu_f\left(\frac{1}{c_1 + c_2}\left(Mgc_2 - hMa\right)\right) - \mu_r\left(\frac{1}{c_1 + c_2}\left(Mgc_1 + hMa\right)\right)$$

or

$$\begin{aligned} a\left(c_1 + c_2\right) &= -\mu_f\left(gc_2 - ha\right) - \mu_r\left(gc_1 + ha\right) \\ &= a\left(h\mu_f - h\mu_r\right) - g\left(c_2\mu_f + c_1\mu_r\right) \end{aligned}$$

or

$$a\left(c_1 + c_2 - h\left(\mu_f - \mu_r\right)\right) = -g\left(c_2\mu_f + c_1\mu_r\right)$$

$$\frac{a}{g} = \frac{-\left(c_2\mu_f + c_1\mu_r\right)}{c_1 + c_2 - h\left(\mu_f - \mu_r\right)}$$

The only way front and rear brakes would lock up at the same time is if the front brake distribution factor is equal to the ratio of the front axle load to the total load, or

$$K_{f\,max} = \frac{F_{x1}}{F_x} = \frac{\mu_f\left(Mgc_2 - hMa\right)}{\mu_f\left(Mgc_2 - hMa\right) + \mu_r\left(Mgc_1 + hMa\right)}$$

Rearranging gives

$$K_{f\,max} = \frac{F_{x1}}{F_x} = \frac{\mu_f\left(c_2 - h\frac{a}{g}\right)}{\mu_f\left(c_2 - h\frac{a}{g}\right) + \mu_r\left(c_1 + h\frac{a}{g}\right)}$$

For Case 1: Front wheels are on a dry surface of $\mu = 0.9$, and the rear wheels are on a wet road of $\mu = 0.2$,

$$\begin{aligned}
\frac{a}{g} &= \frac{-\left(c_2\mu_f + c_1\mu_r\right)}{c_1 + c_2 - h\left(\mu_f - \mu_r\right)} \\
&= \frac{-\left(1.8 \times 0.9 + 1.2 \times 0.2\right)}{3.0 - 0.5 \times \left(0.9 - 0.2\right)} \\
&= \frac{-1.86}{2.65} \\
&= 0.7019
\end{aligned}$$

$$\begin{aligned}
K_{f\,max} &= \frac{F_{x1}}{F_x} \\
&= \frac{\mu_f\left(c_2 - h\frac{a}{g}\right)}{\mu_f\left(c_2 - h\frac{a}{g}\right) + \mu_r\left(c_1 + h\frac{a}{g}\right)} \\
&= \frac{0.9\left(1.8 - 0.5 \times \left(-0.7019\right)\right)}{0.9 \times \left(1.8 - 0.5 \times \left(-0.7019\right)\right) + 0.2\left(1.2 + 0.5 \times \left(-0.7019\right)\right)} \\
&= \frac{1.9359}{1.9359 + 0.1698} = 0.9194
\end{aligned}$$

For Case 2: The rear wheels are on a dry surface of $\mu = 0.9$, and the front wheels are on a wet road of $\mu = 0.2$,

$$\frac{a}{g} = \frac{-\left(1.8 \times 0.2 + 1.2 \times 0.9\right)}{3.0 - 0.5 \times \left(0.2 - 0.9\right)} = -0.4299$$

$$K_{f\,max} = \frac{0.2\left(1.8 - 0.5 \times \left(-0.4299\right)\right)}{0.2 \times \left(1.8 - 0.5 \times \left(-0.4299\right)\right) + 0.9\left(1.2 + 0.5 \times \left(-0.4299\right)\right)} = 0.3125$$

To ensure front wheel lock-up all the time, choose $K_{fmax} = 0.92$.

For Case 1,

Acceleration under front lock-up is

$$a_F = -\frac{\mu\left(\frac{c_2}{c_1+c_2}\right)\left(1+\frac{1-K_f}{K_f}\right)}{\left(-\frac{\mu h}{c_1+c_2}\left(1+\frac{1-K_f}{K_f}\right)+1\right)}g$$

$$= -\frac{0.9\left(\frac{1.8}{3.0}\right)\left(1+\frac{0.08}{0.92}\right)}{\left(-\frac{0.9\times0.5}{3.0}\left(1+\frac{0.08}{0.92}\right)+1\right)}g$$

$$= -0.7013g$$

Acceleration under rear lock-up is

$$a_R = -\frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}g$$

$$= -\frac{0.2\left(\frac{1.2}{3.0}\right)\left(1+\frac{0.92}{0.08}\right)}{\left(\frac{0.2\times0.5}{3.0}\left(1+\frac{0.92}{0.08}\right)+1\right)}g$$

$$= -0.7059g$$

Because $|a_F| < |a_R|$, front lock-up is confirmed to occur first. The vertical load on each axis is

$$F_{z1} = \frac{1}{c_1+c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{1}{3.0}\left(19{,}620\times1.8 - 0.5\times19{,}620\times\left(-0.7013\right)\right)$$

$$= 14{,}065 \text{ N}$$

$$F_{z2} = \frac{1}{c_1+c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{1}{3.0}\left(19{,}620\times1.2 + 0.5\times19{,}620\times\left(-0.7013\right)\right)$$

$$= 5555 \text{ N}$$

The braking force on each axle is

$$F_{x1} = -\mu F_{z1} = -0.9\times14{,}065 = -12{,}659 \text{ N}$$

$$F_{x2} = \frac{1-K_f}{K_f}F_{x1} = -\frac{0.08}{0.92}\times12{,}659 = -1101 \text{ N}$$

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The total braking force is

$$F_x = F_{x1} + F_{x2} = -12,659 - 1101 = -13,760$$

Similar calculations are performed for Case 2. The following table is generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	1.2	1.8	0.5	0.9	19620	0.92	−0.7013	**14065**	**5555**	**−12659**	**−110**	**−13759**	−0.7059
2	1.2	1.8	0.5	0.2	19620	0.92	−0.1353	**12265**	**7405**	**−2443**	**−212**	**−2655**	−0.70588

Note that very little friction force is available to ensure front lock-up when the front wheels are on a slippery surface.

Problem 7.17

Repeat Problem 7.16 with the vehicle loaded. The mass is 2500 kg when loaded with a CG at 1.4 m behind the front axle and 0.6 m above the road.

Solution to Problem 7.17

Similar calculations are made. For K_{fmax}, the following spreadsheet is obtained:

Case	c1	c2	h	μ front	μ rear	Mg	a/g	Kfmax
1	1.4	1.6	0.6	0.9	0.2	24525	−0.6667	0.9000
2	1.4	1.6	0.6	0.2	0.9	24525	−0.462	0.2709

Subsequently, a distribution ratio of 0.9 is selected. This yields the following table:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	1.4	1.6	0.6	0.9	24525	0.9	−0.6667	**16350**	**8175**	**−14715**	**−1635**	**−16350**	**−0.6667**
2	1.4	1.6	0.6	0.2	24525	0.9	−0.1240	**13688**	**10837**	**−2738**	**−304**	**−3042**	**−0.66667**

Problem 7.18

Verify that Eq. 7.47 reduces to Eq. 7.48 when aerodynamic drag, grade, and rolling resistance are ignored.

Solution to Problem 7.18

Equation 7.45 is

$$-\mu\left(\frac{1}{c_1 + c_2}\left(Mgc_1\cos\theta + h_a R_a + hMa_R + Mg\sin\theta\right)\right)\left(1 + \frac{K_f}{1 - K_f}\right)$$
$$- R_a - R_r - Mg\sin(\theta) = Ma_R$$

Ignoring aerodynamic drag means

$$R_a = 0$$

Ignoring grade means

$$\theta = 0$$

Finally, ignoring rolling resistance means

$$R_r = 0$$

Substituting the above equations into Eq. 7.47 yields

$$-\mu\left(\frac{1}{c_1 + c_2}\left(Mgc_1 + hMa_R\right)\right)\left(1 + \frac{K_f}{1 - K_f}\right) = Ma_R$$

Dividing both sides by M and rearranging for a_R yields

$$-\mu\left(\frac{1}{c_1 + c_2}\left(gc_1\right)\right)\left(1 + \frac{K_f}{1 - K_f}\right) = a_R + \frac{\mu h a_R}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right)$$

or

$$-\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)g = a_R\left(1 + \frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right)\right)$$

Further rearranging of terms yields

$$a_R = -\frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)}g \qquad (7.48)$$

Problem 7.19

A passenger-carrying vehicle with a capacity of fewer than 10 passengers (including the driver) is built on a passenger vehicle chassis system. The gross weight of the vehicle is 4000 lb. Determine the minimum braking force required (use Table 7.1), the minimum deceleration required, and the maximum stopping distance obtained without emergency braking. Compare the minimum deceleration required found directly from Table 7.1 with that obtained from the stopping distance requirements.

Solution to Problem 7.19

Assume a speed of 20 mph, or

$$V_i = \frac{20\text{ mi}}{\text{hr}} \times \frac{5280\text{ ft}}{\text{mi}} \times \frac{\text{hr}}{3600\text{ s}} = 29.33\text{ ft/s}$$

from Table 7.1.

The minimum braking force is

$$F_x = (0.652) \times 4000 = 2608 \text{ lb}$$

The acceleration is

$$a = F/M = -\frac{2608 \times 32.2}{4000} = -21.0 \text{ ft/s}^2$$

The time needed to stop is

$$t = \frac{V_f - V_i}{a} = \frac{0 - 29.03}{-21.0} = 1.397 \text{ s}$$

The traveled distance is

$$\begin{aligned} x_f &= x_i + V_i + \frac{1}{2}at^2 \\ &= 0 + 29.03 \times 1.397 - \frac{1}{2} \times 21.0 \times (1.397)^2 \\ &= 20 \text{ ft} \end{aligned}$$

which is the same as that obtained from Table 7.1.

Deceleration is

$$a = -21 \text{ ft/s}^2$$

The minimum distance traveled is

$$x_f = 20 \text{ ft}$$

Problem 7.20

Repeat Problem 7.19 for a property-carrying vehicle weighing 35,544 N (8000 lb). Assume a speed of 20 mph.

Solution to Problem 7.20

$$V_i = \frac{20 \text{ mi}}{\text{hr}} \times \frac{5280 \text{ ft}}{\text{mi}} \times \frac{\text{hr}}{3600 \text{ s}} = 29.33 \text{ ft/s}$$

from Table 7.1.

The minimum braking force is

$$F_x = -(0.528) \times 8000 = 4244 \text{ lb}$$

The acceleration is

$$a = F/M = -\frac{4224 \times 32.2}{8000} = -17.00 \text{ ft/s}^2$$

The time needed to stop is

$$t = \frac{V_f - V_i}{a} = \frac{0 - 29.03}{-21.0} = 1.397 \text{ s}$$

The traveled distance is

$$\begin{aligned} x_f &= x_i + V_i + \frac{1}{2}at^2 \\ &= 0 + 29.03 \times 1.397 - \frac{1}{2} \times 17.0 \times (1.397)^2 \\ &= 23.97 \text{ ft} \end{aligned}$$

which is the same as that obtained from Table 7.1 for acceleration (-17 ft/s^2) and slightly less than the distance requirement of 25 ft.

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Chapter 8

Acceleration

Problem 8.1

An 18,683-N (4200-lb) vehicle is to achieve a maximum speed of 104.6 km/hr (65 mph) in 6 s. If a constant engine torque is assumed, determine the traction force needed between the tires and the road to achieve this speed. Also, find the distance it takes to reach such a speed from an idle condition.

Solution to Problem 8.1

The final velocity can be written as

$$V_f = \frac{104.6 \text{ km}}{\text{hr}} \times \frac{1000 \text{ m}}{\text{km}} \times \frac{\text{hr}}{3600 \text{ s}} = 29.06 \text{ m/s}$$

The acceleration can be found from

$$a = \frac{V_f - V_i}{t} = \frac{29.06 - 0}{6} = 4.84 \text{ m/s}^2$$

The mass of the vehicle is

$$M = 18{,}683/9.81 = 1904 \text{ kg}$$

The force needed to achieve this acceleration is

$$F = M \times a = 2000 \text{ kg} \times 4.63 \text{ m/s}^2 = 9260 \text{ kg m/s}^2 = 9223 \text{ N}$$

The distance needed to reach the maximum speed is

$$X_b = \frac{\left(V_f^2 - V_i^2\right)}{2 \times a} = \frac{\left(29.06\right)^2 - 0}{2 \times \left(4.63\right)} = 91.20 \text{ m}$$

Problem 8.2

Consider a midsize sedan that has a mass of 1500 kg. The wheelbase of the vehicle is 2.1 m and the center of gravity (CG) is 1.0 m from the front axle and 0.5 m above the ground. Determine the load on each axle if the vehicle is under no acceleration, accelerating at 0.6 m/s^2, and accelerating at 1.5 m/s^2.

Solution to Problem 8.2

Given in the problem are the following:

$M = 1500$ kg

$c_1 + c_2 = 2.1$ m

$h = 0.5$ m

$c_1 = 1.0$ m

First, we will find c_2 as

$$c_2 = 2.1 - 1.0 = 1.1 \text{ m}$$

The axle loading under no acceleration (i.e., $a = 0$) is

$$\begin{aligned} F_{z1} &= Mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2}Ma \\ &= 1500 \times 9.81\left(\frac{1.1}{2.1}\right) - \frac{0.5}{2.1} \times 1500 \times 0 \\ &= 7708 \text{ N} \end{aligned}$$

$$\begin{aligned} F_{z2} &= Mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2}Ma \\ &= 1500 \times 9.81\left(\frac{1.0}{2.1}\right) + \frac{0.5}{2.1} \times 1500 \times 0 \\ &= 7007 \text{ N} \end{aligned}$$

Similarly, for $a = 2$ ft/s^2,

$$\begin{aligned} F_{z1} &= Mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2}Ma \\ &= 1500 \times 9.81\left(\frac{1.1}{2.1}\right) - \frac{0.5}{2.1} \times 1500 \times 2 \\ &= 6994 \text{ N} \end{aligned}$$

$$\begin{aligned} F_{z2} &= Mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2}Ma \\ &= 1500 \times 9.81\left(\frac{1.0}{2.1}\right) + \frac{0.5}{2.1} \times 1500 \times 2 \\ &= 7721 \text{ N} \end{aligned}$$

For a = 6 ft/s^2,

$$F_{z1} = Mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2}Ma$$

$$= 1500 \times 9.81\left(\frac{1.1}{2.1}\right) - \frac{0.5}{2.1} \times 1500 \times 6$$

$$= 5566 \text{ N}$$

$$F_{z2} = Mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2}Ma$$

$$= 1500 \times 9.81\left(\frac{1.0}{2.1}\right) + \frac{0.5}{2.1} \times 1500 \times 6$$

$$= 9149 \text{ N}$$

Problem 8.3

A midsize sedan that has a mass of 1500 kg is traveling at 100 km/hr. The front area of the vehicle is 2.0 m^2, and the coefficient of drag is 0.42. The aerodynamic resistance is acting on the CG. The wheelbase of the vehicle is at 2.1 m, and the center of gravity is at 1.0 m from the front axle and at 0.5 m above the ground. The density of air is 1.2256 kg/m^3. What would be the loads if the vehicle were accelerating at 0.6 m/s^2 on a horizontal plane? What would be the axle load when the vehicle goes uphill and downhill at a 6° slope at the same speed?

Solution to Problem 8.3

First, the aerodynamic resistance is calculated as

$$R_a = \frac{\rho}{2}C_D A_F V_r^2$$

$$= \frac{1.2256}{2} \times 0.42 \times 2.0 \times \left(100 \times \frac{1000}{3600}\right)^2$$

$$= 397.2 \text{ N}$$

The forces on the axles (when traveling on a horizontal road) are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2 \cos\theta - h_a R_a - hMa - hMg\sin\theta\right)$$

$$= \frac{1}{c_1 + c_2}\left(Mgc_2 - h_a R_a - hMa\right)$$

$$= \frac{1}{2.1}\left(1500 \times 9.81 \times 1.1 - 0.5 \times 397.2 - 0.5 \times 1500 \times 0.6\right)$$

$$= 7399 \text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1\cos\theta + h_a R_a + hMa + hMg\sin\theta\right)$$
$$= \frac{1}{c_1 + c_2}\left(Mgc_1 + h_a R_a + hMa\right)$$
$$= \frac{1}{2.1}\left(1500 \times 9.81 \times 1.0 + 0.5 \times 397.2 + 0.5 \times 1500 \times 0.6\right)$$
$$= 7316\text{ N}$$

The forces on the axles (when traveling uphill) are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2\cos\theta - h_a R_a - hMa - hMg\sin\theta\right)$$
$$= \frac{1}{2.1}\Big(1500 \times 9.81 \times 1.1 \times \cos(6) - 0.5 \times 397.2 - 0.5$$
$$\times 1500 \times 0.6 - 0.5 \times 1500 \times 9.81 \times \sin(6)\Big)$$
$$= 6991\text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1\cos\theta + h_a R_a + hMa + hMg\sin\theta\right)$$
$$= \frac{1}{2.1}\Big(1500 \times 9.81 \times 1.0 \times \cos(6) + 0.5 \times 397.2 +$$
$$0.5 \times 1500 \times 0.6 + 0.5 \times 1500 \times 9.81 \times \sin(6)\Big)$$
$$= 7644\text{ N}$$

When the vehicle travels downhill, the forces are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2\cos\theta - h_a R_a - hMa - hMg\sin\theta\right)$$
$$= \frac{1}{2.1}\Big(1500 \times 9.81 \times 1.1 \times \cos(6) - 0.5 \times 397.2 - 0.5 \times 1500$$
$$\times 0.6 + 0.5 \times 1500 \times 9.81 \times \sin(6)\Big)$$
$$= 7723\text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1\cos\theta + h_a R_a + hMa + hMg\sin\theta\right)$$
$$= \frac{1}{2.1}\Big(1500 \times 9.81 \times 1.0 \times \cos(6) + 0.5 \times 397.2 + 0.5$$
$$\times 1500 \times 0.6 - 0.5 \times 1500 \times 9.81 \times \sin(6)\Big)$$
$$= 6911\text{ N}$$

Problem 8.4

A 22,215-N (5000-lb) passenger car has a 2743.2-mm (108-in.) wheelbase and a CG at 1066.8 mm (42 in.) behind the front axle and at 457.2 mm (18 in.) above the ground. The tractive effort distribution gives the front axle 90% of the total tractive force. The car is moving on a horizontal plane. Determine which tires will skid first if the car is moving on a surface with first, a coefficient of friction of 0.85 and, second, a coefficient of friction of 0.15. Ignore drag and rolling resistance.

Solution to Problem 8.4

$$M = 5000/32.2 = 155.28 \text{ slugs}$$

$$c_1 + c_2 = 108 \text{ in.}$$

$$c_1 = 42 \text{ in.}$$

$$c_2 = 66 \text{ in.}$$

$$K_f = 0.9$$

$$h = 18$$

$$\theta = 0$$

$$R_a = 0$$

$$R_r = 0$$

For the first surface where $\mu = 0.85$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.85\left(\frac{66}{108}\right)\left(1 + \frac{0.1}{0.9}\right)}{\left(\frac{0.85 \times 18}{108}\left(1 + \frac{0.1}{0.9}\right) + 1\right)} g$$

$$= \frac{0.5772}{1.1575} g$$

$$= 0.4987g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g$$

$$= \frac{0.85\left(\frac{42}{108}\right)\left(1 + \frac{0.9}{0.1}\right)}{\left(-\frac{0.85 \times 18}{108}\left(1 + \frac{0.9}{0.1}\right) + 1\right)} g$$

$$= \frac{3.3056}{-0.4167} g$$

$$= -7.9334g$$

Note here that the negative sign indicates that under the given conditions, the rear tires will never skid with positive acceleration. Thus, the front tires will skid first.

For the second surface where $\mu = 0.15$,

Acceleration under the front skid is

$$a_F = \frac{0.15\left(\frac{66}{108}\right)\left(1+\frac{0.1}{0.9}\right)}{\left(\frac{0.15\times 18}{108}\left(1+\frac{0.1}{0.9}\right)+1\right)}g = 0.09910g$$

Acceleration under the rear skid is

$$a_R = \frac{0.15\left(\frac{42}{108}\right)\left(1+\frac{0.9}{0.1}\right)}{\left(-\frac{0.15\times 18}{108}\left(1+\frac{0.9}{0.1}\right)+1\right)}g = 0.7778g$$

Because $a_F < a_R$, the front tires will skid first.

Problem 8.5

Consider the previous problem. The car is moving on a horizontal plane, and drag and rolling resistance are negligible. Determine the maximum tractive force, the maximum acceleration, and the distribution factor needed to achieve them. The coefficient of friction is 0.25.

Solution to Problem 8.5

$$M = 5000/32.2 = 155.28 \text{ slugs}$$

$c_1 + c_2 = 108$ in.

$c_1 = 42$ in.

$c_2 = 66$ in.

$K_f = 0.9$

$H = 18$

$\theta = 0$

$R_a = 0$

$R_r = 0$

The front brake distribution factor needed to achieve the maximum braking force is

$$K_f = \frac{(c_2 + \mu h)}{(c_1 + c_2)} = \frac{66 - 0.25\times 18}{108} = 0.5694$$

The maximum braking force F_x is

$$F_x = \mu Mg\cos(\theta) = 0.25 \times 5000 = 1250 \text{ lb}$$

The maximum acceleration is

$$a = \frac{\mu Mg\cos(\theta) - R_a - R_r - Mg\sin(\theta)}{M} = \mu g = 8.05 \text{ ft/s}^2$$

Problem 8.6

Consider Problem 8.5. A vehicle strategy requires a front axle skid on a dry surface, and the front torque distribution factor is chosen at 0.7. The vehicle is traveling on a low-friction surface with a coefficient of friction of 0.25. Determine the maximum tractive force, and identify which axle would skid first.

Solution to Problem 8.6

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.25\left(\frac{66}{108}\right)\left(1 + \frac{0.3}{0.7}\right)}{\left(\frac{0.25 \times 18}{108}\left(1 + \frac{0.3}{0.7}\right) + 1\right)} g$$

$$= 0.2060g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g$$

$$= \frac{0.25\left(\frac{42}{108}\right)\left(1 + \frac{0.7}{0.3}\right)}{\left(-\frac{0.25 \times 18}{108}\left(1 + \frac{0.7}{0.3}\right) + 1\right)} g$$

$$= 0.3763g$$

Because $a_F < a_R$, the front skid occurs first. The normal forces are

$$
\begin{aligned}
F_{z1} &= \frac{1}{c_1 + c_2}\left(Mgc_2 - hMa_F\right) \\
&= \frac{Mg}{c_1 + c_2}\left(c_2 - h\frac{a_F}{g}\right) \\
&= \frac{5000}{108}\left(66 - 18 \times (0.1859)\right) \\
&= 2884 \text{ lb}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2} &= \frac{1}{c_1 + c_2}\left(Mgc_1 + hMa_F\right) \\
&= \frac{Mg}{c_1 + c_2}\left(c_1 + h\frac{a_F}{g}\right) \\
&= \frac{5000}{108}\left(42 + 18 \times (0.1859)\right) \\
&= 2116 \text{ lb}
\end{aligned}
$$

Note that

$$F_{z1} + F_{z2} = 5000 \text{ lb}$$

$$F_{x1} = \mu F_{z1} = 0.25 \times 2883 = 721.0 \text{ lb}$$

$$F_{x2} = \frac{1 - K_f}{K_f} F_{x1} = \frac{1 - 0.7}{0.7} \times (721.0) = 309.0 \text{ lb}$$

The total friction force is

$$F_x = F_{x1} + F_{x2} = 1030 \text{ lb}$$

Problem 8.7

A vehicle has a mass of 1600 kg when lightly loaded (driver only), with a CG at 0.8 m behind the front axle and at 0.45 m above the road. When loaded, the vehicle has a mass of 2000 kg, with a CG at 1.2 m behind the front axle and at 0.5 in. above the road. The wheelbase is 2.6 m. The vehicle is to achieve maximum possible tractive force and front skid under the following severe conditions:

Vehicle is loaded, on a dry surface of $\mu = 0.85$

Vehicle is loaded, on a wet surface of $\mu = 0.15$

Vehicle is lightly loaded, on a dry surface of $\mu = 0.85$

Vehicle is lightly loaded, on a wet road of $\mu = 0.15$

Determine the value of K_f that you recommend, given that the drive system your company installs has a constant torque distribution value (no feedback control system). Determine the axle loads and the maximum tractive force.

Solution to Problem 8.7

For Case 1: Vehicle is lightly loaded, on a dry surface of $\mu = 0.85$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.8 - 0.85 \times 0.45}{2.6} = 0.5452$$

For Case 2: Vehicle is lightly loaded, on a slippery surface of $\mu = 0.15$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.8 - 0.15 \times 0.45}{2.6} = 0.6663$$

For Case 3: Vehicle is loaded, on a dry surface of $\mu = 0.85$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.4 - 0.15 \times 0.5}{2.6} = 0.375$$

For Case 4: Vehicle is loaded, on a slippery surface of $\mu = 0.15$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.4 - 0.15 \times 0.5}{2.6} = 0.5096$$

To ensure front lock-up, a distribution factor of 0.7 is selected.

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.85$,

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.85\left(\frac{1.8}{2.6}\right)\left(1 + \frac{0.3}{0.7}\right)}{\left(\frac{0.85 \times 0.5}{2.6}\left(1 + \frac{0.3}{0.7}\right) + 1\right)} g$$

$$= 0.6947g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}g$$

$$= \frac{0.85\left(\frac{0.8}{2.6}\right)\left(1+\frac{0.7}{0.3}\right)}{\left(-\frac{0.85\times 0.5}{2.6}\left(1+\frac{0.7}{0.3}\right)+1\right)}g$$

$$= 1.7107g$$

Because $a_F < a_R$, the front skid occurs first. The normal forces are

$$F_{z1} = \frac{1}{c_1+c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{Mg}{c_1+c_2}\left(c_2 - h\frac{a_F}{g}\right)$$

$$= \frac{15,696}{2.6}\left(1.8 - 0.5\times(0.6947)\right)$$

$$= 8770\text{ N}$$

$$F_{z2} = \frac{1}{c_1+c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{Mg}{c_1+c_2}\left(c_1 + h\frac{a_F}{g}\right)$$

$$= \frac{15,696}{2.6}\left(0.8 + 0.5\times(0.6947)\right)$$

$$= 6926\text{ N}$$

Note that

$$F_{z1} + F_{z2} = 15,696\text{ N}$$

Because the front tires skid first,

$$F_{x1} = \mu F_{z1} = 0.85\times 8979 = 7632\text{ N}$$

$$F_{x2} = \frac{1-K_f}{K_f}F_{x1} = \frac{1-0.7}{0.7}\times(7632) = 3271\text{ N}$$

The total friction force is

$$F_x = F_{x1} + F_{x2} = 10,903\text{ N}$$

Similar calculations are performed for all remaining cases. For all other cases, the following table is generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	0.8	1.8	0.45	0.85	15696	0.7	0.6947	8979	6717	7632	3271	10903	1.7107
2	0.8	1.8	0.45	0.15	15696	0.7	0.1430	10478	5218	1572	674	2245	0.1684
3	1.2	1.4	0.5	0.85	19620	0.7	0.5301	8565	11055	7280	3120	10400	2.8732
4	1.2	1.4	0.5	0.15	19620	0.7	0.1108	10146	9474	1522	652	2174	0.2553

Problem 8.8

A V-6 gasoline engine is putting a maximum of 300 hp of gross power at 5000 rpm. Air induction losses are found to be 2.8% at that engine speed. The hot and cold end exhaust losses are found to be 8%. In addition, losses due to air conditioning, power steering, the alternator, and other accessories add up to 15 hp under specific driving conditions. The ambient temperature is 35°C, and dry pressure is 105 kPa. Determine the power available to the transmission.

Solution to Problem 8.8

Total losses of the induction and exhaust systems are

$$2.8\% + 8 = 10.8\%$$

The power available to the accessories and transmission is

$$89.2\% \times 300 \text{ hp} = 267.6 \text{ hp}$$

The power available to the transmission under standard conditions is

$$267.6 - 15 = 252.6 \text{ hp}$$

The power ratio under the given conditions is

$$\frac{P}{P_o} = \frac{1}{1.18\left(\left(\frac{99}{B_d}\right)\left(\frac{T+273}{298}\right)^{1/2}\right) - 0.18}$$

$$= \frac{1}{1.18\left(\left(\frac{99}{105}\right)\left(\frac{35+273}{298}\right)^{1/2}\right) - 0.18}$$

$$= 1.0514$$

The horsepower available to the transmission is

$$1.0514 \times 252.6 \text{ hp} = 265.6 \text{ hp}$$

Problem 8.9

An engine delivers 325 N · m (240 ft · lb) torque at 2500 rpm engine speed. It is coupled to a torque converter and an automatic speed transmission. Determine the output speed and the output torque of the converter of the characteristics given in Figure 8.22. Determine the efficiency of the torque converter.

Solution to Problem 8.9

The engine speed and torque are

$$T_e = 325\ \mathrm{N \cdot m}$$

$$n_e = 2500\ \mathrm{rpm}$$

The capacity factor of the torque converter is the same as that of the engine.

$$K_{tc} = K_e = \frac{\omega_e}{\sqrt{T_e}} = \frac{2500}{\sqrt{325}} = 138.7\frac{\mathrm{rpm}}{\sqrt{\mathrm{Nm}}}$$

Use a graph similar to Figure 8.22 to determine the speed and torque ratio.

The speed ratio is 0.78, and the torque ratio is 1.18. Therefore,

$$n_{tc} = 0.78 \times 2500 = 1950\ \mathrm{rpm}$$

$$T_{tc} = 1.18 \times 300 = 354\ \mathrm{Nm}$$

With the use of Eq. 8.49, the efficiency is

$$\eta_{tc} = \frac{P_{out}}{P_{in}} = \frac{T_{tcout}\,\omega_{tcout}}{T_{tcin}\,\omega_{tcin}} = \frac{1950 \times 354}{2500 \times 300} = 0.92$$

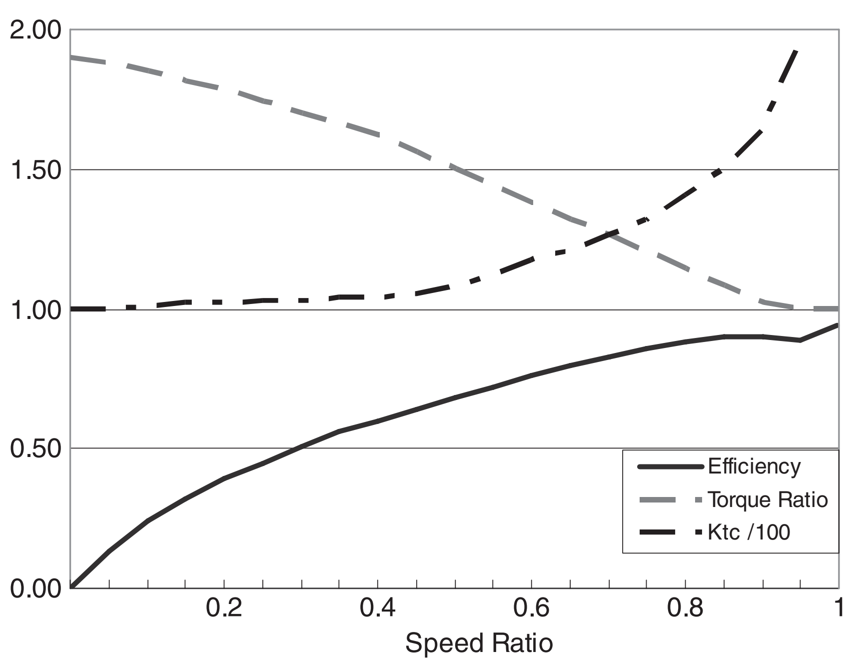

Figure 8.9 Efficiency and torque ratios as functions of speed ratio for a torque converter.

Problem 8.10

A vehicle (including wheels) weighs 30,000 N. Each wheel weighs 300 N, and the rolling radius of the wheels is 0.35 m; the radius of gyration is 0.28 m. The engine is driven at 3000 rpm and is delivering 400 N · m. The total drivetrain efficiency is 84%, and the transmission is in fourth gear with a ratio of 3.8 to 1. The components rotating at engine speed have an inertia of 1.0 kg · m^2. Assume a slip of the running gear at 3% and a coefficient of rolling resistance of 2.5%. The front area of the vehicle is 2.2 m^2, the aerodynamic drag coefficient is 0.38, and the speed of the wind is zero. Determine the acceleration at that point.

Solution to Problem 8.10

Given

$$M = 30{,}000/9.8 = 3058 \text{ kg}$$

$$m_w = 300/9.81 = 30.58 \text{ kg}$$

$$r = 0.35 \text{ m}$$

$$r_g = 0.28 \text{ m}$$

$$n_e = 3000 \text{ rpm}$$

$$T_e = 400 \text{ N·m}$$

$$\eta = 0.84$$

$$\xi_t = 3.8$$

$$\sum I = 1.0 \text{ kg·m}^2$$

$$i = 0.03$$

$$\mu_r = 0.025$$

$$C_D = 0.38$$

$$A_F = 2.2 \text{ m}^2$$

1. Calculate the mass factor (γ_m).

$$I_w = m_w\left(r_g\right)^2 = 30.58\left(0.28\right)^2 = 2.397\,\text{kg}\cdot\text{m}^2$$

$$\gamma_m = 1 + \frac{1}{Mr^2}\left\{\sum I_w + \sum I_1\xi_1^2 + \sum I_2\xi_2^2 + \cdots + \sum I_n\xi_n^2\right\}$$

$$= 1 + \frac{1}{3058 \times \left(0.35\right)^2}\left\{4 \times 2.397 + 1.0 \times 3.8^2\right\} = 1.0641$$

2. Calculate the tractive force (Eq. 8.47).

$$F_x = \frac{T_e\xi_o\eta_t}{r} = \frac{400 \times 3.8 \times 0.86}{0.35} = 3735 \text{ N}$$

3. Calculate the vehicle speed (Eq. 8.48).

$$
\begin{aligned}
V &= \frac{n_e r(1-i)}{\xi_o} \\
&= \frac{3000 \times (2\pi) \times 0.35 \times (1-0.03)}{3.8} \\
&= 1684 \text{ m/min} \\
&= 101.044 \text{ km/hr}
\end{aligned}
$$

Because the wind is at zero speed,

$$V_r = V = 101.044 \text{ km/hr} = 28.07 \text{ m/s}$$

4. Calculate the rolling resistance force (Eq. 7.26).

$$R_f = \mu_r mg = 0.025 \times 30{,}000 = 750 \text{ N}$$

5. Calculate the aerodynamic drag force (Eq. 7.27). Assume that the density of air is 1.2256 kg/m^3.

$$R_a = \frac{\rho}{2} C_D A_F V_r^2 = \frac{1.2256}{2} \times 0.38 \times 2.2 \times (28.07)^2 = 403.7 \text{ N}$$

Note the unit conversion factors.

6. Calculate the acceleration (Eq. 8.53).

$$a = \frac{F_{net}}{\gamma_m m} = \frac{F - \sum R}{\gamma_m m} = \frac{3735 - 750 - 404}{1.0641 \times 3058} = 0.7932 \text{ m/s}^2$$

Problem 8.11

Consider Problem 8.10. Assume that the engine torque remained constant during take-off acceleration. Also, acceleration is occurring while the transmission is engaged in the same gear as that found in Problem 8.10. Determine how long it would take the vehicle to reach a constant speed of 100 km/hr. Assume a wind speed of 20 km/hr against the vehicle direction.

Solution to Problem 8.11

1. Unit conversion is

$$V_f = 100 \frac{\text{km}}{\text{hr}} = 100 \times \frac{1000 \text{ m}}{3600 \text{ s}} = 27.78 \text{ m/s}$$

2. Determine the drag (Eq. 7.27).

$$R_a = \frac{\rho}{2} C_D A_F V_r^2 = \frac{1.2256}{2} \times 0.38 \times 2.2 \times (V_r)^2 = 0.5123\, V_r^2$$

3. Determine F_{net}.

$$F_{net} = F - \sum R = 3735 - 750 - 0.5123V_r^2 = 2985 - 0.5123V_r^2$$

4. Determine the time (Eq. 8.57).

$$t = \gamma_m m \int_{V_i}^{V_f} \frac{dV}{F_{net}} = 1.0641 \times 3058 \times \int_0^{27.78} \frac{dV}{2985 - 0.5123V_r^2} = 31.7 \text{ s}$$

Problem 8.12

Consider a light truck that weighs 20,000 N under a certain load. The wheelbase of the vehicle is at 3.0 m, and the CG is at 1.3 m from the front axle and at 0.75 m above the ground. Determine the load on each axle under the maximum traction limited acceleration with a coefficient of friction of 0.8.

Solution to Problem 8.12

The optimal distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.7 - 0.8 \times 0.75}{3.0} = 0.3667$$

Acceleration under the front skid is

$$\begin{aligned} a_F &= \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g \\ &= \frac{0.80\left(\frac{1.7}{3.0}\right)\left(1 + \frac{0.6333}{0.3667}\right)}{\left(\frac{0.80 \times 0.75}{3.0}\left(1 + \frac{0.6333}{0.3667}\right) + 1\right)} g \\ &= 0.8000g \end{aligned}$$

Acceleration under the rear skid is

$$\begin{aligned} a_R &= \frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g \\ &= \frac{0.80\left(\frac{1.3}{3.0}\right)\left(1 + \frac{0.3667}{0.6333}\right)}{\left(-\frac{0.80 \times 0.75}{3.0}\left(1 + \frac{0.3667}{0.6333}\right) + 1\right)} g \\ &= 0.8000g \end{aligned}$$

Because $a_F = a_R$, all wheels skid at the same time. The normal forces are

$$\begin{aligned} F_{z1} &= \frac{1}{c_1 + c_2}\left(Mgc_2 - hMa_F\right) \\ &= \frac{Mg}{c_1 + c_2}\left(c_2 - h\frac{a_F}{g}\right) \\ &= \frac{20{,}000}{3.0}\left(1.7 - 0.5 \times \left(0.8\right)\right) \\ &= 8667 \text{ N} \end{aligned}$$

$$\begin{aligned} F_{z2} &= \frac{1}{c_1 + c_2}\left(Mgc_1 + hMa_F\right) \\ &= \frac{Mg}{c_1 + c_2}\left(c_1 + h\frac{a_F}{g}\right) \\ &= \frac{20000}{3.0}\left(1.3 + 0.5 \times \left(0.8\right)\right) \\ &= 11333 \text{ N} \end{aligned}$$

Note that

$$F_{z1} + F_{z2} = 20{,}000 \text{ N}$$

In addition,

$$F_{x1} = \mu F_{z1} = 0.80 \times 8667 = 6933 \text{ N}$$

$$F_{x2} = \frac{1 - K_f}{K_f} F_{x1} = \frac{1 - 0.3667}{0.3667} \times \left(6933\right) = 11975 \text{ N}$$

The total friction force is

$$F_x = F_{x1} + F_{x2} = 18907 \text{ N}$$

Problem 8.13

A 1300-kg compact passenger car has a 2.0-m wheelbase and a CG at 0.96 m behind the front axle and at 0.6 m above the ground. The car is traveling on a horizontal plane. Determine which tires will skid first if the car is moving on a surface with a coefficient of friction of 0.9, and if the front axle force distribution factors are 0.4, 0.6, and 0.8.

Solution to Problem 8.13

For Case 1: $K_f = 0.4$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.90\left(\frac{1.04}{2.0}\right)\left(1 + \frac{0.6}{0.4}\right)}{\left(\frac{0.90 \times 0.6}{2.0}\left(1 + \frac{0.6}{0.4}\right) + 1\right)} g$$

$$= 0.6985g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g$$

$$= \frac{0.90\left(\frac{0.96}{2.0}\right)\left(1 + \frac{0.4}{0.6}\right)}{\left(-\frac{0.90 \times 0.6}{2.0}\left(1 + \frac{0.4}{0.6}\right) + 1\right)} g$$

$$= 1.3091g$$

Because $a_F < a_R$, the front tires will skid first.

For Case 2: $K_f = 0.6$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.90\left(\frac{1.04}{2.0}\right)\left(1 + \frac{0.4}{0.6}\right)}{\left(\frac{0.90 \times 0.6}{2.0}\left(1 + \frac{0.4}{0.6}\right) + 1\right)} g$$

$$= 0.5379g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(-\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}g$$

$$= \frac{0.90\left(\frac{0.96}{2.0}\right)\left(1+\frac{0.6}{0.4}\right)}{\left(-\frac{0.90\times 0.6}{2.0}\left(1+\frac{0.6}{0.4}\right)+1\right)}g$$

$$= 3.3231g$$

Because $a_F < a_R$, the front tires will skid first.

For Case 3: $K_f = 0.8$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1+c_2}\right)\left(1+\frac{1-K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1+c_2}\left(1+\frac{1-K_f}{K_f}\right)+1\right)}g$$

$$= \frac{0.90\left(\frac{1.04}{2.0}\right)\left(1+\frac{0.2}{0.8}\right)}{\left(\frac{0.90\times 0.6}{2.0}\left(1+\frac{0.2}{0.8}\right)+1\right)}g$$

$$= 0.4374g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(-\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}g$$

$$= \frac{0.90\left(\frac{0.96}{2.0}\right)\left(1+\frac{0.8}{0.2}\right)}{\left(-\frac{0.90\times 0.6}{2.0}\left(1+\frac{0.8}{0.2}\right)+1\right)}g$$

$$= -6.1714g$$

Note here that the negative sign indicates that under the given conditions, the rear tires will never skid with positive acceleration. Thus, the front tires will skid first.

Problem 8.14

Consider a vehicle with a 1500-kg mass when lightly loaded (driver only), with a CG at 1.0 m behind the front axle and at 0.45 m above the road. The mass is 2200 kg when loaded with a CG at 1.25 m behind the front axle and at 0.5 m above the road. The wheelbase is 2.7 m. The vehicle is to achieve the maximum possible tractive force and front skid under the following severe conditions:

Vehicle is loaded, on a dry surface of $\mu = 0.9$

Vehicle is loaded, on a slippery surface of $\mu = 0.2$

Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$

Vehicle is lightly loaded, on a slippery road of $\mu = 0.2$

Determine the value of the front force distribution factor K_f that you recommend, given that the force system has a constant value. Determine the axle loads and the maximum tractive force.

Solution to Problem 8.14

For Case 1: Vehicle is loaded, dry surface of $\mu = 0.9$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.45 - 0.9 \times 0.5}{2.7} = 0.3704$$

For Case 2: Vehicle is loaded, on a slippery surface of $\mu = 0.2$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.45 - 0.2 \times 0.5}{2.7} = 0.5000$$

For Case 3: Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.7 - 0.9 \times 0.45}{2.7} = 0.4796$$

For Case 4: Vehicle is lightly loaded, on a slippery road of $\mu = 0.2$,

The optimal force distribution factor is

$$K_{f\,max} = \frac{(c_2 - \mu h)}{(c_1 + c_2)} = \frac{1.8 - 0.2 \times 0.45}{2.7} = 0.6333$$

To ensure front lock-up, a distribution factor of 0.6 is selected.

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.9\left(\frac{1.45}{2.7}\right)\left(1 + \frac{0.4}{0.6}\right)}{\left(\frac{0.9 \times 0.5}{2.7}\left(1 + \frac{0.4}{0.6}\right) + 1\right)} g$$

$$= 0.6304g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)} g$$

$$= \frac{0.9\left(\frac{1.25}{2.7}\right)\left(1 + \frac{0.6}{0.4}\right)}{\left(-\frac{0.9 \times 0.5}{2.7}\left(1 + \frac{0.6}{0.4}\right) + 1\right)} g$$

$$= 1.7857g$$

Because $a_F < a_R$, the front skid occurs first. The normal forces are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{Mg}{c_1 + c_2}\left(c_2 - h\frac{a_F}{g}\right)$$

$$= \frac{21{,}560}{2.7}\left(1.45 - 0.5 \times (0.6304)\right)$$

$$= 9061 \text{ N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{Mg}{c_1 + c_2}\left(c_1 + h\frac{a_F}{g}\right)$$

$$= \frac{21560}{2.7}\left(1.25 + 0.5 \times (0.6304)\right)$$

$$= 12{,}499 \text{ N}$$

Because the front tires skid first,

$$F_{x1} = \mu F_{z1} = 0.90 \times 9061 = 8155 \text{ N}$$

$$F_{x2} = \frac{1 - K_f}{K_f} F_{x1} = \frac{1 - 0.6}{0.6} \times (9061) = 6041 \text{ N}$$

The total friction force is

$$F_x = F_{x1} + F_{x2} = 14196 \text{ N}$$

Similar calculations are performed for all remaining cases. For all other cases, the following table is generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	1.25	1.45	0.5	0.9	21560	0.6	0.6304	9061	12499	8155	5437	**13592**	1.7857
2	1.25	1.45	0.5	0.2	21560	0.6	0.1686	10905	10655	2181	1454	**3635**	0.2551
3	1	1.7	0.45	0.9	14700	0.6	0.7556	7404	7296	6664	4443	**11107**	1.3333
4	1	1.7	0.45	0.2	14700	0.6	0.1988	8768	5932	1754	1169	**2923**	0.2020

Problem 8.15

The vehicle described in Problem 8.14 has a traction control system that allows control of each wheel force distribution. Determine the maximum tractive force achieved under the following conditions:

Vehicle is loaded, on a dry surface of $\mu = 0.9$

Vehicle is loaded, on a slippery surface of $\mu = 0.2$

Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$

Vehicle is lightly loaded, on a slippery road of $\mu = 0.2$

Solution to Problem 8.15

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\frac{c_2}{c_1 + c_2}\right)\left(1 + \frac{1 - K_f}{K_f}\right)}{\left(\frac{\mu h}{c_1 + c_2}\left(1 + \frac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.9\left(\frac{1.45}{2.7}\right)\left(1 + \frac{1.0 - 0.3704}{0.3704}\right)}{\left(\frac{0.9 \times 0.5}{2.7}\left(1 + \frac{1.0 - 0.3704}{0.3704}\right) + 1\right)} g$$

$$= 0.90g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1+c_2}\right)\left(1+\frac{K_f}{1-K_f}\right)}{\left(-\frac{\mu h}{c_1+c_2}\left(1+\frac{K_f}{1-K_f}\right)+1\right)}$$

$$= \frac{0.9\left(\frac{1.25}{2.7}\right)\left(1+\frac{0.3704}{1.0-0.3704}\right)}{\left(-\frac{0.9\times 0.5}{2.7}\left(1+\frac{0.3704}{1.0-0.3704}\right)+1\right)}g$$

$$= 0.90g$$

Because $a_F = a_R$, all wheels will skid at the same time. The normal forces are

$$F_{z1} = \frac{1}{c_1+c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{Mg}{c_1+c_2}\left(c_2 - h\frac{a_F}{g}\right)$$

$$= \frac{21,560}{2.7}\left(1.45 - 0.5\times\left(0.90\right)\right)$$

$$= 7985 \text{ N}$$

$$F_{z2} = \frac{1}{c_1+c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{Mg}{c_1+c_2}\left(c_1 + h\frac{a_F}{g}\right)$$

$$= \frac{21560}{2.7}\left(1.25 + 0.5\times\left(0.90\right)\right)$$

$$= 13,575 \text{ N}$$

Because the front tires skid first,

$$F_{x1} = \mu F_{z1} = 0.90 \times 7985 = 7187 \text{ N}$$

$$F_{x2} = \frac{1-K_f}{K_f}F_{x1} = \frac{1.0-0.3704}{0.3704}\times\left(7187\right) = 12,217 \text{ N}$$

The total friction force is

$$F_x = F_{x1} + F_{x2} = 19,404 \text{ N}$$

Similar calculations are performed for all remaining cases. For all other cases, the following table is generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	1.25	1.45	0.5	0.9	21560	0.3704	0.9000	7985	13575	7187	12217	**19404**	0.9000
2	1.25	1.45	0.5	0.2	21560	0.5	0.2000	10780	10780	2156	2156	**4312**	0.2000
3	1	1.7	0.45	0.9	14700	0.4796	0.9000	7051	7649	6346	6885	**13230**	0.9000
4	1	1.7	0.45	0.2	14700	0.5963	0.2000	8766	5934	1753	1187	**2940**	0.2000

Problem 8.16

For the vehicle in Problems 8.14 and 8.15, determine the maximum tractive force if the vehicle is moving on a split-mue surface with the right wheels on a 0.2 coefficient of friction surface and the left wheels on a 0.8 coefficient of friction surface.

Solution to Problem 8.16

Assume total symmetry for the right and left wheels. We will use the two loading conditions (loaded and lightly loaded), with a load distribution factor of 0.6.

Note that for a split-mue condition, under front lock-up,

$$F_{x1} = \mu_L \frac{F_{z1}}{2} + \mu_R \frac{F_{z1}}{2} = \left(\frac{\mu_L + \mu_R}{2}\right) F_{z1}$$

Thus, the effective friction is

$$\mu = \left(\frac{\mu_L + \mu_R}{2}\right) = 0.5$$

For Case 1: Vehicle is loaded, on a dry surface of $\mu = 0.9$,

Acceleration under the front skid is

$$a_F = \frac{\mu\left(\dfrac{c_2}{c_1 + c_2}\right)\left(1 + \dfrac{1 - K_f}{K_f}\right)}{\left(\dfrac{\mu h}{c_1 + c_2}\left(1 + \dfrac{1 - K_f}{K_f}\right) + 1\right)} g$$

$$= \frac{0.5\left(\dfrac{1.45}{2.7}\right)\left(1 + \dfrac{0.4}{0.6}\right)}{\left(\dfrac{0.5 \times 0.5}{2.7}\left(1 + \dfrac{0.4}{0.6}\right) + 1\right)} g$$

$$= 0.3877g$$

Acceleration under the rear skid is

$$a_R = \frac{\mu\left(\frac{c_1}{c_1 + c_2}\right)\left(1 + \frac{K_f}{1 - K_f}\right)}{\left(-\frac{\mu h}{c_1 + c_2}\left(1 + \frac{K_f}{1 - K_f}\right) + 1\right)}$$

$$= \frac{0.5\left(\frac{1.25}{2.7}\right)\left(1 + \frac{0.6}{0.4}\right)}{\left(-\frac{0.5 \times 0.5}{2.7}\left(1 + \frac{0.6}{0.4}\right) + 1\right)} g$$

$$= 0.7530g$$

Because $a_F < a_R$, the front skid occurs first. The normal forces are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2 - hMa_F\right)$$

$$= \frac{Mg}{c_1 + c_2}\left(c_2 - h\frac{a_F}{g}\right)$$

$$= \frac{21560}{2.7}\left(1.45 - 0.5 \times \left(0.3877\right)\right)$$

$$= 10{,}031\ \text{N}$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1 + hMa_F\right)$$

$$= \frac{Mg}{c_1 + c_2}\left(c_1 + h\frac{a_F}{g}\right)$$

$$= \frac{21560}{2.7}\left(1.25 + 0.5 \times \left(0.3877\right)\right)$$

$$= 11{,}529\ \text{N}$$

Because the front tires skid first,

$$F_{x1} = \mu F_{z1} = 0.50 \times 10031 = 5015\ \text{N}$$

$$F_{x2} = \frac{1 - K_f}{K_f} F_{x1} = \frac{1 - 0.6}{0.6} \times \left(5015\right) = 3344\ \text{N}$$

The total friction force is

$$F_x = F_{x1} + F_{x2} = 8359\ \text{N}$$

Similar calculations are performed for all other cases. The following table is generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fx1	Fx2	Fx	aR/g
1	1.25	1.45	0.5	0.5	21560	0.6	0.3877	10031	11529	5015	3344	**8359**	0.7530
2	1	1.7	0.45	0.5	14700	0.6	0.4607	8127	6573	4063	2709	**6772**	0.5848

Problem 8.17

For Problem 8.16, assume that control of the tractive force on each wheel is possible under an advanced traction control system. Determine the maximum tractive force that can be applied.

Solution to Problem 8.17

For Case 1: Vehicle is loaded,

$$F_x = \mu F_z = 0.5 \times 21560 = 10,780 \text{ N}$$

For Case 2: Vehicle is lightly loaded,

$$F_x = \mu F_z = 0.5 \times 1470 = -735 \text{ N}$$

Problem 8.18

Derive Eq. 8.17.

Solution to Problem 8.18

The maximum tractive force applied in a rear-wheel-drive system is $F_{x2} = M \times a_R$. When the grade, rolling resistance, and aerodynamic drag are considered, Eq. 8.15, $F_{x2} = \mu F_{z2}$ is substituted into Eq. 8.9. This yields the maximum tractive forces as

$$F_{x2} = \frac{\mu}{c_1 + c_2}\left(Mgc_1 \cos\theta + h\left(F_x - R_r\right)\right)$$

Note that $F_x = F_{x2}$ for rear-wheel-drive vehicles. Substituting ($R_r = \mu r\, Mg$) into the preceding equation and rewriting it for F_x gives

$$F_{x2} - \frac{\mu h}{c_1 + c_2} F_{x2} = \frac{\mu Mg}{c_1 + c_2}\left(c_1 \cos\theta - h\mu_r\right)$$

Rearranging the terms yields

$$F_{x2} = \frac{\dfrac{\mu Mg}{c_1 + c_2}\left(c_1 \cos\theta - h\mu_r\right)}{1 - \dfrac{\mu h}{c_1 + c_2}} \tag{8.17}$$

Problem 8.19

Equations 8.40 and 8.41 give the maximum angle that a rear-wheel-drive vehicle or a front-wheel-drive vehicle can climb at a constant speed. Derive a similar equation for an all-wheel-drive vehicle.

Solution to Problem 8.19

Equation 8.7 can be rewritten as

$$F_x = R_f + R_a + Ma + Mg\sin\theta$$

Also,

$$F_x = F_{x1} + F_{x2}$$

Under front skid condition, we have

$$F_{x1} = \mu F_{z1}$$

$$F_{x2} = \frac{1-K_f}{K_f}F_{x1} = \frac{1-K_f}{K_f}\times \mu F_{z1}$$

Subsequently,

$$F_x = \mu F_{z1} + \frac{1-K_f}{K_f}\times \mu F_{z1} = \left(1+\frac{1-K_f}{K_f}\right)F_{z1}$$

Equation 8.13 yields

$$\begin{aligned} F_{z1} &= \frac{1}{c_1+c_2}\left(Mgc_2\cos\theta - h\left(F_{x1}-R_r\right)\right) \\ &= \frac{1}{c_1+c_2}\left(Mgc_2\cos\theta - h\left(\mu F_{z1}-R_r\right)\right) \\ &= \frac{1}{c_1+c_2}\left(Mgc_2\cos\theta + hR_r\right) - \frac{h\mu}{c_1+c_2}F_{z1} \end{aligned}$$

Rearranging the preceding equation yields

$$F_{z1} = \frac{\frac{1}{c_1+c_2}\left(Mgc_2\cos\theta + hR_r\right)}{1+\frac{h\mu}{c_1+c_2}}$$

Multiplying by the sum $c_1 + c_2$ yields

$$F_{z1} = \frac{Mgc_2 \cos\theta + hR_r}{c_1 + c_2 + h\mu}$$

Substituting for F_x yields

$$F_x = \mu F_{z1} + \frac{1 - K_f}{K_f} \times \mu F_{z1}$$

$$= \left(1 + \frac{1 - K_f}{K_f}\right) \frac{Mgc_2 \cos\theta + hR_r}{c_1 + c_2 + h\mu}$$

Setting $a = 0$, $R_a = 0$, and $R_r = \mu rMg$ in Eq. 8.7 and equating it to the preceding equation yields

$$F_x = R_r + R_a + Ma + Mg \sin\theta$$

$$\mu_r Mg + Mg \sin(\theta_{max}) = \left(1 + \frac{1 - K_f}{K_f}\right) \frac{Mgc_2 \cos(\theta_{max}) + \mu_r hMg}{c_1 + c_2 + h\mu}$$

or

$$\sin(\theta_{max}) = \left(1 + \frac{1 - K_f}{K_f}\right) \frac{c_2 \cos(\theta_{max}) + h\mu_r}{c_1 + c_2 + h\mu} - \mu_r$$

A similar derivation can be made with a rear skid assumption.

Problem 8.20

Consider a vehicle with a CG at 1.000 m behind the front axle and at 0.460 m above the road. The rolling resistance is 0.02, and the wheelbase is 2.750 m. Find the maximum angle that can be withstood without losing traction for both front-wheel drive and rear-wheel drive when the coefficient of friction is 1.0.

Solution to Problem 8.20

The following are given:

$$c_1 + c_2 = 2.750 \text{ m}$$

$$c_1 = 1.000 \text{ m}$$

$$\mu_r = 0.02$$

$$h = 0.46 \text{ m}$$

$$\mu = 1.0$$

For rear-wheel-drive vehicles,

First trial, $\theta_{max} = 0$

$$\sin(\theta_{max}) = \frac{\frac{\mu}{c_1 + c_2}(c_2 \cos\theta_{max} - h\mu_r)}{1 - \frac{\mu h}{c_1 + c_2}} - \mu_r$$

$$= \frac{\frac{1.0}{2.75}(1.75 \times \cos(0) - 0.46 \times 0.02)}{1 - \frac{1.0 \times 0.46}{2.75}} - 0.02$$

$$= 0.74$$

Second trial, $\theta_{max} = \sin^{-1}(0.74 \text{ rad}) = 0.83 \text{ rad}$

$$\sin(\theta_{max}) = \frac{\frac{1.0}{2.75}(1.75 \times \cos(0.83\text{rad}) - 0.46 \times 0.02)}{1 - \frac{1.0 \times 0.46}{2.75}} - 0.02 = 0.49$$

Third trial, $\theta_{max} = \sin^{-1}(0.49 \text{ rad}) = 0.51 \text{ rad}$

$$\sin(\theta_{max}) = \frac{\frac{1.0}{2.75}(1.75 \times \cos(0.51 \text{ rad}) - 0.46 \times 0.02)}{1 - \frac{1.0 \times 0.46}{2.75}} - 0.02 = 0.56$$

Fourth trial, $\theta_{max} = \sin^{-1}(0.56 \text{ rad}) = 0.60 \text{ rad}$

After more trials, $\theta_{max} = 0.63 \text{ rad} = 36°$

For front-wheel-drive vehicles,

First trial, $\theta_{max} = 0$

$$\sin(\theta_{max}) = \frac{\frac{\mu}{c_1 + c_2}(c_2 \cos\theta_{max} + h\mu_r)}{1 + \frac{\mu h}{c_1 + c_2}} + \mu_r$$

$$= \frac{\frac{1.0}{2.75}(1.75 \times \cos(0) + 0.46 \times 0.02)}{1 + \frac{1.0 \times 0.46}{2.75}} + 0.02$$

$$= 0.57$$

Second trial, $\theta_{max} = \sin^{-1}(0.57\text{ rad}) = 0.60\text{ rad}$

$$\sin(\theta_{max}) = \frac{\frac{1.0}{2.75}\left(1.75\times\cos(0.60\text{rad}) + 0.46\times 0.02\right)}{1+\frac{1.0\times 0.46}{2.75}} + 0.02$$

$$= 0.47$$

Third trial, $\theta_{max} = \sin^{-1}(0.47\text{ rad}) = 0.49\text{ rad}$

$$\sin(\theta_{max}) = \frac{\frac{1.0}{2.75}\left(1.75\times\cos(0.49\text{ rad}) + 0.46\times 0.02\right)}{1+\frac{1.0\times 0.46}{2.75}} + 0.02$$

$$= 0.50$$

Fourth trial, $\theta_{max} = \sin^{-1}(0.50\text{ rad}) = 0.53\text{ rad}$

After more trials, $\theta_{max} = 0.52\text{ rad} = 30°$

Note that for higher friction, rear-wheel-drive vehicles have better capability of climbing slopes.

Chapter 9

Total Vehicle Dynamics

Problem 9.1

Consider a vehicle 20,000 N under a certain load. The wheelbase of the vehicle is at 3.00 m, and the center of gravity is at 1.20 m from the front axle and at 0.6 m above the ground. The speed of the vehicle is 100 km/hr, the turning radius is 180 m, the wheel tread is 1.25 m, $K_\phi = 10{,}000\ \text{N}\cdot\text{m}$, the roll center height is 0.45 m, and the roll angle is 5°. Determine the load on each wheel (F_{z1i}, F_{z1o}, F_{z2i}, F_{z2o}) if the vehicle is under no acceleration and is decelerating at 3.00 m/s^2.

Solution to Problem 9.1

The following are given:

$W = 20{,}000$ N (thus, $M = 20{,}000/9.8 = 2040$ kg)

$c_1 + c_2 = 3.0$ m

$h = 0.6$ m

$c_1 = 1.20$ m (thus, $c_2 = 3.0 - 1.20 = 1.8$ m)

$V = 100$ km/hr (27.78 m/s)

$R = 180$ m

$t = 1.25$ m

$K_\phi = 10{,}000\ \text{N}\cdot\text{m}$

$h_r = 0.45$ m

$\phi = 5°$ (0.08727 radians)

Under no acceleration (i.e., where a = 0),

$$F_{z1} = mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2} ma$$

$$= 20{,}000\left(\frac{1.8}{3.0}\right) - \frac{0.6}{3.0} \times \frac{20000}{9.8} \times 0$$

$$= 12{,}000 \text{ N}$$

$$F_{y1} = F_{z1}\frac{V^2}{2Rg}$$

$$= 12{,}000 \times \frac{(27.78)^2}{2 \times 180 \times 9.81}$$

$$= 2622 \text{ N}$$

$$\Delta F_{z1} = \frac{F_{y1}h_r + K_\varphi \varphi}{t}$$

$$= \frac{2622 \times 0.45 + 10{,}000 \times 0.08727}{1.25}$$

$$= 1642 \text{ N}$$

$$F_{z1i} = \frac{F_{z1}}{2} - \Delta F_{z1}$$

$$= \frac{12{,}000}{2} - 1642$$

$$= 4358 \text{ N}$$

$$F_{z1o} = \frac{F_{z1}}{2} + \Delta F_{z1}$$

$$= \frac{12{,}000}{2} + 1642$$

$$= 7642 \text{ N}$$

Similarly, for the rear axle,

$$F_{z2} = mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2} ma$$

$$= 20{,}000\left(\frac{1.2}{3.0}\right) + \frac{1.8}{3.0} \times \frac{20{,}000}{9.8} \times 0$$

$$= 8000 \text{ N}$$

$$F_{y2} = F_{z2}\frac{V^2}{2Rg}$$

$$= 8000 \times \frac{(27.78)^2}{2 \times 180 \times 9.8}$$

$$= 1748 \text{ N}$$

$$\Delta F_{z2} = \frac{F_{y2}h_r + K_\varphi \varphi}{t}$$

$$= \frac{1748 \times 0.45 + 10{,}000 \times 0.08727}{1.25}$$

$$= 1327\ \text{N}$$

$$F_{z2i} = \frac{F_{z2}}{2} - \Delta F_{z2}$$

$$= \frac{8000}{2} - 1327$$

$$= 2673\ \text{N}$$

$$F_{z2o} = \frac{F_{z2}}{2} + \Delta F_{z2}$$

$$= \frac{8000}{2} + 1327$$

$$= 5327\ \text{N}$$

For the case where $a = -3\ \text{m/s}^2$,

$$F_{z1} = mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2} ma$$

$$= 20{,}000\left(\frac{1.8}{3.0}\right) - \frac{0.6}{3.0} \times \frac{20{,}000}{9.8} \times -3.0$$

$$= 13{,}223\ \text{N}$$

$$F_{y1} = F_{z1}\frac{V^2}{2Rg}$$

$$= 13{,}223 \times \frac{(27.78)^2}{2 \times 180 \times 9.81}$$

$$= 2889\ \text{N}$$

$$\Delta F_{z1} = \frac{F_{y1}h_r + K_\varphi \varphi}{t}$$

$$= \frac{2889 \times 0.45 + 10{,}000 \times 0.08727}{1.25}$$

$$= 1738\ \text{N}$$

$$F_{z1i} = \frac{F_{z1}}{2} - \Delta F_{z1}$$

$$= \frac{13{,}223}{2} - 1738$$

$$= 4873\ \text{N}$$

$$F_{z1o} = \frac{F_{z1}}{2} + \Delta F_{z1}$$

$$= \frac{13{,}223}{2} + 1738$$

$$= 8350\ \text{N}$$

Similarly, for the rear axle,

$$
\begin{aligned}
F_{z2} &= mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2} ma \\
&= 20{,}000\left(\frac{1.2}{3.0}\right) + \frac{0.6}{3.0} \times \frac{20000}{9.8} \times -3 \\
&= 6777 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
F_{y2} &= F_{z2} \frac{V^2}{2Rg} \\
&= 6777 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} \\
&= 1481 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
\Delta F_{z2} &= \frac{F_{y2} h_r + K_{\varphi}\varphi}{t} \\
&= \frac{1481 \times 0.45 + 10{,}000 \times 0.08727}{1.25} \\
&= 1231 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2i} &= \frac{F_{z2}}{2} - \Delta F_{z2} \\
&= \frac{6777}{2} - 1231 \\
&= 2157 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2o} &= \frac{F_{z2}}{2} + \Delta F_{z2} \\
&= \frac{6777}{2} + 1231 \\
&= 4620 \text{ N}
\end{aligned}
$$

Problem 9.2

Find the wheel forces for Problem 9.1 if the vehicle accelerates at a rate of 1.00 m/s^2.

Solution to Problem 9.2

$$
\begin{aligned}
F_{z1} &= mg\left(\frac{c_2}{c_1 + c_2}\right) - \frac{h}{c_1 + c_2} ma \\
&= 20{,}000\left(\frac{1.8}{3.0}\right) - \frac{0.6}{3.0} \times \frac{20000}{9.8} \times 1.0 \\
&= 11{,}592 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
F_{y1} &= F_{z1}\frac{V^2}{2Rg} \\
&= 11{,}592 \times \frac{(27.78)^2}{2 \times 180 \times 9.81} \\
&= 2533\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
\Delta F_{z1} &= \frac{F_{y1}h_r + K_\varphi \varphi}{t} \\
&= \frac{2533 \times 0.45 + 10{,}000 \times 0.08727}{1.25} \\
&= 1610\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z1i} &= \frac{F_{z1}}{2} - \Delta F_{z1} \\
&= \frac{11{,}592}{2} - 1610 \\
&= 4186\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z1o} &= \frac{F_{z1}}{2} + \Delta F_{z1} \\
&= \frac{11{,}592}{2} + 1610 \\
&= 7406\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2} &= mg\left(\frac{c_1}{c_1 + c_2}\right) + \frac{h}{c_1 + c_2}ma \\
&= 20{,}000\left(\frac{1.2}{3.0}\right) + \frac{0.6}{3.0} \times \frac{20{,}000}{9.8} \times 1.0 \\
&= 8408\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
F_{y2} &= F_{z2}\frac{V^2}{2Rg} \\
&= 8408 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} \\
&= 1837\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
\Delta F_{z2} &= \frac{F_{y2}h_r + K_\varphi \varphi}{t} \\
&= \frac{8408 \times 0.45 + 10{,}000 \times 0.08727}{1.25} \\
&= 1359\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2i} &= \frac{F_{z2}}{2} - \Delta F_{z2} \\
&= \frac{8408}{2} - 1359 \\
&= 2844\ \text{N}
\end{aligned}
$$

$$
\begin{aligned}
F_{z2o} &= \frac{F_{z2}}{2} + \Delta F_{z2} \\
&= \frac{8408}{2} + 1359 \\
&= 5563\ \text{N}
\end{aligned}
$$

Problem 9.3

A 2500-kg sports car has a 2.5-m wheel base with a CG at 1.0 m behind the front axle and at 0.35 m above the ground. The braking effort distribution gives the front axle 60% of the total braking force. The car is moving on a horizontal plane, and drag and rolling resistance are ignored. The car is going around a turning radius of 150 m at 120 km/hr. The wheel tread is 1.40 m, $K_\phi = 12{,}000\ \text{N}\cdot\text{m}$, the roll center height is 0.4 m, and the roll angle is 8°. Determine which tires will lock up first if the car is moving on a surface with first, a coefficient of friction of 0.8, and second, a coefficient of friction of 0.2.

Solution to Problem 9.3

For the first surface, where $\mu = 0.8$,

Acceleration under front lock-up is

$$
a_F = \frac{-\dfrac{2\mu}{K_f}\left[\left(\dfrac{Mgc_2}{c_1 + c_2}\right)\left(\dfrac{1}{2} - \dfrac{V^2 h_r}{2Rgt}\right) - \dfrac{K_\varphi \varphi}{t}\right]}{M\left[1 - \dfrac{\mu h}{K_f\left(c_1 + c_2\right)}\left(1 - \dfrac{V^2 h_r}{Rgt}\right)\right]}
$$

$$
a_F = \frac{-\dfrac{2\times 0.8}{0.6}\left[\left(\dfrac{2500\times 9.81\times 1.5}{2.5}\right)\left(\dfrac{1}{2} - \dfrac{(33.33)^2\times 0.4}{2\times 150\times 9.81\times 1.4}\right) - \dfrac{12{,}000\times 0.1396}{1.4}\right]}{2500\left[1 - \dfrac{0.8\times 0.35}{0.6\times 2.5}\left(1 - \dfrac{(33.33)^2\times 0.4}{150\times 9.81\times 1.4}\right)\right]}
$$

$$= -0.5826g$$

$$
a_R = \frac{-\dfrac{2\mu}{1 - K_f}\left(\left\{\dfrac{Mgc_1}{c_1 + c_2}\right\}\left(\dfrac{1}{2} - \dfrac{V^2 h_r}{2Rgt}\right) - \dfrac{K_\varphi \varphi}{t}\right)}{M\left[1 + \dfrac{\mu h}{\left(1 - K_f\right)\left(c_1 + c_2\right)}\left(1 - \dfrac{V^2 h_r}{Rgt}\right)\right]}
$$

$$a_R = \frac{-\frac{2\times 0.8}{0.4}\left[\left(\frac{2500\times 9.81\times 1.0}{2.5}\right)\left(\frac{1}{2}-\frac{(33.33)^2\times 0.4}{2\times 150\times 9.81\times 1.4}\right)-\frac{12{,}000\times 0.1396}{1.4}\right]}{2500\left[1+\frac{0.8\times 0.35}{0.4\times 2.5}\left(1-\frac{(33.33)^2\times 0.4}{150\times 9.81\times 1.4}\right)\right]}$$

$$= -0.3544g$$

Because $|a_F| > |a_R|$, the rear tires will lock up first.

For the second surface, where $\mu = 0.2$,

Acceleration under front lock-up is

$$a_F = \frac{-\frac{2\times 0.2}{0.6}\left[\left(\frac{2500\times 9.81\times 1.5}{2.5}\right)\left(\frac{1}{2}-\frac{(33.33)^2\times 0.4}{2\times 150\times 9.81\times 1.4}\right)-\frac{12{,}000\times 0.1396}{1.4}\right]}{2500\left[1-\frac{0.2\times 0.35}{0.6\times 2.5}\left(1-\frac{(33.33)^2\times 0.4}{150\times 9.81\times 1.4}\right)\right]}$$

$$= -0.1290g$$

$$a_R = \frac{-\frac{2\times 0.2}{0.4}\left[\left(\frac{2500\times 9.81\times 1.0}{2.5}\right)\left(\frac{1}{2}-\frac{(33.33)^2\times 0.4}{2\times 150\times 9.81\times 1.4}\right)-\frac{12{,}000\times 0.1396}{1.4}\right]}{2500\left[1+\frac{0.2\times 0.35}{0.4\times 2.5}\left(1-\frac{(33.33)^2\times 0.4}{150\times 9.81\times 1.4}\right)\right]}$$

$$= -0.1024g$$

Because $|a_F| > |a_R|$, the rear tires will lock up first.

Problem 9.4

Consider the vehicle in Problem 9.3. The torque distribution gives the front axle 60% of the total torque delivered to the front wheels. Determine which tires will skid first if the car is moving on a surface with first, a coefficient of friction of 0.8, and second, a coefficient of friction of 0.2.

Solution to Problem 9.4

For the first surface, where $\mu = 0.80$,

Acceleration under the front skid is

$$a_F = \frac{\frac{2\times 0.8}{0.6}\left[\left(\frac{2500\times 9.81\times 1.5}{2.5}\right)\left(\frac{1}{2}-\frac{(33.33)^2\times 0.4}{2\times 150\times 9.81\times 1.4}\right)-\frac{12{,}000\times 0.1396}{1.4}\right]}{2500\left[1+\frac{0.8\times 0.35}{0.6\times 2.5}\left(1-\frac{(33.33)^2\times 0.4}{150\times 9.81\times 1.4}\right)\right]}$$

$$= 0.4338g$$

Acceleration under the rear skid is

$$a_R = \frac{\frac{2 \times 0.8}{0.4}\left[\left(\frac{2500 \times 9.81 \times 1.0}{2.5}\right)\left(\frac{1}{2} - \frac{(33.33)^2 \times 0.4}{2 \times 150 \times 9.81 \times 1.4}\right) - \frac{12{,}000 \times 0.1396}{1.4}\right]}{2500\left[1 - \frac{0.8 \times 0.35}{0.4 \times 2.5}\left(1 - \frac{(33.33)^2 \times 0.4}{150 \times 9.81 \times 1.4}\right)\right]}$$

$$= 0.5538g$$

Because $|a_F| < |a_R|$, the front tires will skid first.

For the second surface, where $\mu = 0.2$,

Acceleration under the front skid is

$$a_F = \frac{\frac{2 \times 0.2}{0.6}\left[\left(\frac{2500 \times 9.81 \times 1.5}{2.5}\right)\left(\frac{1}{2} - \frac{(33.33)^2 \times 0.4}{2 \times 150 \times 9.81 \times 1.4}\right) - \frac{12{,}000 \times 0.1396}{1.4}\right]}{2500\left[1 + \frac{0.2 \times 0.35}{0.6 \times 2.5}\left(1 - \frac{(33.33)^2 \times 0.4}{150 \times 9.81 \times 1.4}\right)\right]}$$

$$= 0.1199g$$

Acceleration under the rear skid is

$$a_R = \frac{\frac{2 \times 0.2}{0.4}\left[\left(\frac{2500 \times 9.81 \times 1.5}{2.5}\right)\left(\frac{1}{2} - \frac{(33.33)^2 \times 0.4}{2 \times 150 \times 9.81 \times 1.4}\right) - \frac{12{,}000 \times 0.1396}{1.4}\right]}{2500\left[1 - \frac{0.2 \times 0.35}{0.4 \times 2.5}\left(1 - \frac{(33.33)^2 \times 0.4}{150 \times 9.81 \times 1.4}\right)\right]}$$

$$= 0.1143g$$

Because $|a_F| > |a_R|$, the rear tires will skid first.

Problem 9.5

Derive the acceleration and tractive force expressions for the skid of front-wheel-drive (FWD) systems going around a corner.

Solution to Problem 9.5

For a front-wheel-drive vehicle, $K_f = 1.0$. Thus, Eq. 9.18 becomes

$$a_F = \frac{2\mu\left(\left\{\frac{1}{c_1 + c_2}\left(Mgc_2\cos\theta - h_aR_a - hMg\sin\theta\right)\right\}\left(\frac{1}{2} - \frac{V^2h_r}{2Rgt}\right) - \frac{K_\varphi\varphi}{t}\right) - R_a - R_r - Mg\sin(\theta)}{M\left[1 + \frac{\mu h}{(c_1 + c_2)}\left(1 - \frac{V^2h_r}{Rgt}\right)\right]}$$

The maximum tractive force applied in front-wheel-drive systems is $F_{x1} = M \times a_F$. (Note that $F_{x2} = 0$.)

Problem 9.6

Derive the acceleration and tractive force expressions for skid of rear-wheel-drive systems going around a corner.

Solution to Problem 9.6

For a rear-wheel-drive vehicle, $K_f = 0$. Thus, Eq. 9.22 becomes

$$a_R = \frac{\dfrac{2\mu}{1}\left(\left\{\dfrac{1}{c_1 + c_2}\left(Mgc_2\cos\theta + h_aR_a + hMg\sin\theta\right)\right\}\left(\dfrac{1}{2} - \dfrac{V^2h_r}{2Rgt}\right) - \dfrac{K_\varphi\varphi}{t}\right) - R_a - R_r - Mg\sin(\theta)}{M\left[1 - \dfrac{\mu h}{(c_1 + c_2)}\left(1 - \dfrac{V^2h_r}{Rgt}\right)\right]}$$

The maximum tractive force applied in rear-wheel-drive systems is $F_{x2} = M \times a_R$. (Note that $F_{x1} = 0$.)

Problem 9.7

Derive an expression for the brake force distribution factor, so that lock-up occurs simultaneously at both front and rear wheels.

Solution to Problem 9.7

One can follow the method presented for deriving Eq. 7.53 in Chapter 7 of *Road Vehicle Dynamics* (see *Road Vehicle Dynamics*, by Dukkipati et al., SAE International, Warrendale, PA, ISBN 978-0-7680-1643-7). Alternatively, one can find the optimal distribution factor by making the acceleration derived for front lock-up and that derived for rear lock-up equal. We will ignore grade, aerodynamic factors, and other resistance. Thus, we will make Eq. 9.11 equal to Eq. 9.15:

$$\frac{-\dfrac{2\mu}{K_f}\left[\left(\dfrac{Mgc_2}{c_1 + c_2}\right)\left(\dfrac{1}{2} - \dfrac{V^2h_r}{2Rgt}\right) - \dfrac{K_\varphi\varphi}{t}\right]}{M\left[1 - \dfrac{\mu h}{K_f(c_1 + c_2)}\left(1 - \dfrac{V^2h_r}{Rgt}\right)\right]}$$

$$= \frac{-\dfrac{2\mu}{1 - K_f}\left(\left\{\dfrac{Mgc_1}{c_1 + c_2}\right\}\left(\dfrac{1}{2} - \dfrac{V^2h_r}{2Rgt}\right) - \dfrac{K_\varphi\varphi}{t}\right)}{M\left[1 + \dfrac{\mu h}{(1 - K_f)(c_1 + c_2)}\left(1 - \dfrac{V^2h_r}{Rgt}\right)\right]}$$

Multiplying the above equation by $-M/2\mu$ yields

$$\frac{\frac{1}{K_f}\left[\left(\frac{Mgc_2}{c_1+c_2}\right)\left(\frac{1}{2}-\frac{V^2h_r}{2Rgt}\right)-\frac{K_\varphi\varphi}{t}\right]}{\left[1-\frac{\mu h}{K_f\left(c_1+c_2\right)}\left(1-\frac{V^2h_r}{Rgt}\right)\right]}=\frac{\frac{1}{1-K_f}\left(\left\{\frac{Mgc_1}{c_1+c_2}\right\}\left(\frac{1}{2}-\frac{V^2h_r}{2Rgt}\right)-\frac{K_\varphi\varphi}{t}\right)}{\left[1+\frac{\mu h}{\left(1-K_f\right)\left(c_1+c_2\right)}\left(1-\frac{V^2h_r}{Rgt}\right)\right]}$$

Further manipulation yields

$$\left[\left(\frac{Mgc_2}{c_1+c_2}\right)\left(\frac{1}{2}-\frac{V^2h_r}{2Rgt}\right)-\frac{K_\varphi\varphi}{t}\right]\left[1+\frac{\mu h}{\left(1-K_f\right)\left(c_1+c_2\right)}\left(1-\frac{V^2h_r}{Rgt}\right)\right]$$
$$=\frac{K_f}{1-K_f}\left[1-\frac{\mu h}{K_f\left(c_1+c_2\right)}\left(1-\frac{V^2h_r}{Rgt}\right)\right]\left(\left\{\frac{Mgc_1}{c_1+c_2}\right\}\left(\frac{1}{2}-\frac{V^2h_r}{2Rgt}\right)-\frac{K_\varphi\varphi}{t}\right)$$

Let

$$K_1=\frac{V^2h_r}{2Rgt}$$
$$K_2=\frac{K_\varphi\varphi}{t}$$
$$c=c_1+c_2$$

Then, the above equation becomes

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]\left[1+\frac{\mu h}{\left(1-K_f\right)c}\left(1-K_1\right)\right]$$
$$=\frac{K_f}{1-K_f}\left[1-\frac{\mu h}{K_fc}\left(1-K_1\right)\right]\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2}-K_1\right)-K_2\right)$$

This can be simplified to

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]\left[1+\frac{\mu h}{\left(1-K_f\right)c}\left(1-K_1\right)\right]$$
$$=\left[\frac{K_f}{1-K_f}-\frac{1}{1-K_f}\frac{\mu h}{c}\left(1-K_1\right)\right]\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2}-K_1\right)-K_2\right)$$

Multiplying the whole equation by $1 - K_f$ gives

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\left[1 - K_f + \frac{\mu h}{c}\left(1 - K_1\right)\right]$$
$$= \left[K_f - \frac{\mu h}{c}\left(1 - K_1\right)\right]\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$

Expanding the terms yields

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right] - \left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]K_f$$
$$- \left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\frac{\mu h}{c}\left(1 - K_1\right)$$
$$= \left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)K_f$$
$$- \frac{\mu h}{c}\left(1 - K_1\right)\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$

Collecting the terms containing K_f on one side gives

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right] - \left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\frac{\mu h}{c}\left(1 - K_1\right)$$
$$+ \frac{\mu h}{c}\left(1 - K_1\right)\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$
$$= \left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)K_f$$
$$- \left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]K_f$$

or

$$K_f = \frac{\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right] - \left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\frac{\mu h}{c}\left(1 - K_1\right) + \frac{\mu h}{c}\left(1 - K_1\right)\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)}{\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right) - \left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]}$$

The equation can be simplified further.

Problem 9.8

Derive an expression for the torque distribution factor, so that skid occurs simultaneously at both front and rear wheels in all-wheel-drive systems.

Solution to Problem 9.8

We will follow an approach similar to that used in Problem 9.7. We will ignore grade, aerodynamic forces, and other resistance. Thus, we will make Eq. 9.19 equal to Eq. 9.23 as

$$\frac{\dfrac{2\mu}{K_f}\left[\left(\dfrac{Mgc_2}{c_1+c_2}\right)\left(\dfrac{1}{2}-\dfrac{V^2h_r}{2Rgt}\right)-\dfrac{K_\varphi\varphi}{t}\right]}{M\left[1+\dfrac{\mu h}{K_f\left(c_1+c_2\right)}\left(1-\dfrac{V^2h_r}{Rgt}\right)\right]}$$

$$=\frac{\dfrac{2\mu}{1-K_f}\left(\left\{\dfrac{Mgc_1}{c_1+c_2}\right\}\left(\dfrac{1}{2}-\dfrac{V^2h_r}{2Rgt}\right)-\dfrac{K_\varphi\varphi}{t}\right)}{M\left[1-\dfrac{\mu h}{\left(1-K_f\right)\left(c_1+c_2\right)}\left(1-\dfrac{V^2h_r}{Rgt}\right)\right]}$$

or

$$\frac{\dfrac{1}{K_f}\left[\left(\dfrac{Mgc_2}{c_1+c_2}\right)\left(\dfrac{1}{2}-\dfrac{V^2h_r}{2Rgt}\right)-\dfrac{K_\varphi\varphi}{t}\right]}{\left[1+\dfrac{\mu h}{K_f\left(c_1+c_2\right)}\left(1-\dfrac{V^2h_r}{Rgt}\right)\right]}$$

$$=\frac{\dfrac{1}{1-K_f}\left(\left\{\dfrac{Mgc_1}{c_1+c_2}\right\}\left(\dfrac{1}{2}-\dfrac{V^2h_r}{2Rgt}\right)-\dfrac{K_\varphi\varphi}{t}\right)}{1\left[1-\dfrac{\mu h}{\left(1-K_f\right)\left(c_1+c_2\right)}\left(1-\dfrac{V^2h_r}{Rgt}\right)\right]}$$

Further manipulation yields

$$\left[\left(\frac{Mgc_2}{c_1+c_2}\right)\left(\frac{1}{2}-\frac{V^2h_r}{2Rgt}\right)-\frac{K_\varphi\varphi}{t}\right]\left[1-\frac{\mu h}{\left(1-K_f\right)\left(c_1+c_2\right)}\left(1-\frac{V^2h_r}{Rgt}\right)\right]$$

$$=\frac{K_f}{1-K_f}\left(\left\{\frac{Mgc_1}{c_1+c_2}\right\}\left(\frac{1}{2}-\frac{V^2h_r}{2Rgt}\right)-\frac{K_\varphi\varphi}{t}\right)\left[1+\frac{\mu h}{K_f\left(c_1+c_2\right)}\left(1-\frac{V^2h_r}{Rgt}\right)\right]$$

Let

$$\begin{aligned} K_1 &= \frac{V^2 h_r}{2Rgt} \\ K_2 &= \frac{K_\varphi \varphi}{t} \\ c &= c_1 + c_2 \end{aligned}$$

The previous equation becomes

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\left[1 - \frac{\mu h}{\left(1 - K_f\right)c}\left(1 - K_1\right)\right]$$
$$= \frac{K_f}{1 - K_f}\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)\left[1 + \frac{\mu h}{K_f c}\left(1 - K_1\right)\right]$$

or

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\left[1 - \frac{\mu h}{\left(1 - K_f\right)c}\left(1 - K_1\right)\right]$$
$$= \frac{1}{1 - K_f}\left[K_f + \frac{\mu h}{c}\left(1 - K_1\right)\right]\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$

Multiplying the whole equation by $1 - K_f$ gives

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]\left[1 - K_f - \frac{\mu h}{c}\left(1 - K_1\right)\right]$$
$$= \left[K_f + \frac{\mu h}{c}\left(1 - K_1\right)\right]\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$

Expanding the terms gives

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right] - K_f\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]$$
$$- \frac{\mu h}{c}\left(1 - K_1\right)\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2} - K_1\right) - K_2\right]$$
$$= K_f\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$
$$+ \frac{\mu h}{c}\left(1 - K_1\right)\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2} - K_1\right) - K_2\right)$$

Gathering any terms with K_f gives

$$\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]-\frac{\mu h}{c}(1-K_1)\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]$$
$$-\frac{\mu h}{c}(1-K_1)\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2}-K_1\right)-K_2\right)$$
$$=K_f\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2}-K_1\right)-K_2\right)$$
$$+K_f\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]$$

or

$$K_f=\frac{\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]-\frac{\mu h}{c}(1-K_1)\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]-\frac{\mu h}{c}(1-K_1)\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2}-K_1\right)-K_2\right)}{\left(\left\{\frac{Mgc_1}{c}\right\}\left(\frac{1}{2}-K_1\right)-K_2\right)+\left[\left(\frac{Mgc_2}{c}\right)\left(\frac{1}{2}-K_1\right)-K_2\right]}$$

Problem 9.9

Consider the vehicle in Problem 9.1. The vehicle is traveling 100 km/hr up a hill that has a 6° slope. The front area of the vehicle is 2.0 m^2, and the coefficient of drag is 0.42. The aerodynamic resistance is acting on the CG, and the density of air is 1.2256 kg/m^3. Determine the load on each axle if the vehicle is under no acceleration. What would be the loads if the vehicle is accelerating at 0.6 m/s^2? What would be these forces when the vehicle is traveling at the same speed but down a hill that has a 6° slope?

Solution to Problem 9.9

The following are given in the problem:

$W = 20{,}000$ N (thus, $M = 20{,}000/9.8 = 2040$ kg)

$c_1 + c_2 = 3.0$ m

$h = 0.6$ m

$c_1 = 1.20$ m (thus, $c_2 = 3.0 - 1.20 = 1.8$ m)

$V = 100$ km/hr (27.78 m/s)

$R = 180$ m

$t = 1.25$ m

$K_\phi = 10{,}000\ N\cdot m$

$h_r = 0.45$ m

$\phi = 5°$ (0.08727 radians)

$\theta = 6°$

$A_F = 2.0\ m^2$

$C_D = 0.42$

$\rho = 1.2256\ kg/m^3$

First, the aerodynamic resistance is calculated as

$$R_a = \frac{\rho}{2} C_D A_F V_r^2$$
$$= \frac{1.2256}{2} \times 0.42 \times 2.0 \times \left(100 \times \frac{1000}{3600}\right)^2 = 397.2\ N$$

The vehicle travels up a hill that has a 6° slope.

For Condition 1: Under no acceleration,

The forces on the axles (when going uphill) are

$$F_{z1} = \frac{1}{c_1 + c_2}\left(Mgc_2 \cos\theta - h_a R_a - hMa - hMg \sin\theta\right)$$
$$= \frac{1}{3.0}\Big(20{,}000 \times 1.8 \times \cos(6) - 0.6 \times 397.2 - 0.6$$
$$\times 2039 \times 0 - 0.5 \times 20000 \times \sin(6)\Big)$$
$$= 11437\ N$$

$$F_{y1} = F_{z1}\frac{V^2}{2Rg} = 11437 \times \frac{(27.78)^2}{2 \times 180 \times 9.81} = 2499\ N$$

$$\Delta F_{z1} = \frac{F_{y1}h_r + K_\varphi \varphi}{t}$$
$$= \frac{2499 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1598\ N$$

$$F_{z1i} = \frac{F_{z1}}{2} - \Delta F_{z1} = \frac{11{,}437}{2} - 1598 = 4121\ N$$

$$F_{z1o} = \frac{F_{z1}}{2} + \Delta F_{z1} = \frac{11{,}437}{2} + 1598 = 7316\ N$$

$$F_{z2} = \frac{1}{c_1 + c_2}\left(Mgc_1 \cos\theta + h_a R_a + hMa + hMg \sin\theta\right)$$
$$= \frac{1}{3.0}\Big(20{,}000 \times 1.2 \times \cos(6) + 0.6 \times 397.2$$
$$+ 0.6 \times 2039 \times 0 + 0.6 \times 20000 \times \sin(6)\Big)$$
$$= 8454\ N$$

$$F_{y2} = F_{z2}\frac{V^2}{2Rg} = 8454 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} = 1847\ \text{N}$$

$$\begin{aligned}\Delta F_{z2} &= \frac{F_{y2}h_r + K_\varphi \varphi}{t} \\ &= \frac{1847 \times 0.45 + 10{,}000 \times 0.08727}{1.25} \\ &= 1363\ \text{N}\end{aligned}$$

$$F_{z2i} = \frac{F_{z2}}{2} - \Delta F_{z2} = \frac{8454}{2} - 1363 = 2863\ \text{N}$$

$$F_{z2o} = \frac{F_{z2}}{2} + \Delta F_{z2} = \frac{8454}{2} + 1363 = 5590\ \text{N}$$

For Condition 2: If the vehicle is accelerating at 0.6 m/s^2,

The forces on the axles (when traveling uphill) are

$$\begin{aligned}F_{z1} &= \frac{1}{3.0}\big(20{,}000 \times 1.8 \times \cos(6) - 0.6 \times 397.2 - 0.6 \\ &\qquad \times 2039 \times 0.6 - 0.5 \times 20{,}000 \times \sin(6)\big) \\ &= 11{,}192\ \text{N}\end{aligned}$$

$$F_{y1} = 11{,}192 \times \frac{(27.78)^2}{2 \times 180 \times 9.81} = 2445\ \text{N}$$

$$\Delta F_{z1} = \frac{2445 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1578\ \text{N}$$

$$F_{z1i} = \frac{11{,}192}{2} - 1578 = 4018\ \text{N}$$

$$F_{z1o} = \frac{11{,}192}{2} + 1578 = 7174\ \text{N}$$

$$\begin{aligned}F_{z2} &= \frac{1}{3.0}\big(20{,}000 \times 1.2 \times \cos(6) + 0.6 \times 397.2 + 0.6 \\ &\qquad \times 2039 \times 0.6 + 0.6 \times 20000 \times \sin(6)\big) \\ &= 8698\ \text{N}\end{aligned}$$

$$F_{y2} = 8698 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} = 1900\ \text{N}$$

$$\Delta F_{z2} = \frac{1900 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1382\ \text{N}$$

$$F_{z2i} = \frac{8698}{2} - 1382 = 2967 \text{ N}$$

$$F_{z2o} = \frac{8698}{2} + 1382 = 5731 \text{ N}$$

The vehicle goes down a hill that has a 6° slope.

For Condition 3: Under no acceleration,

The forces on the axles (when traveling uphill) are

$$\begin{aligned} F_{z1} &= \frac{1}{3.0}\big(20{,}000 \times 1.8 \times \cos(-6) - 0.6 \times 397.2 - 0.6 \\ &\qquad \times 2039 \times 0 - 0.5 \times 20000 \times \sin(-6)\big) \\ &= 12{,}273 \text{ N} \end{aligned}$$

$$F_{y1} = 12{,}273 \times \frac{(27.78)^2}{2 \times 180 \times 9.81} = 2681 \text{ N}$$

$$\Delta F_{z1} = \frac{2499 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1663 \text{ N}$$

$$F_{z1i} = \frac{12{,}273}{2} - 1663 = 4473 \text{ N}$$

$$F_{z1o} = \frac{12{,}273}{2} + 1598 = 7800 \text{ N}$$

$$\begin{aligned} F_{z2} &= \frac{1}{3.0}\big(20{,}000 \times 1.2 \times \cos(-6) + 0.6 \times 397.2 + 0.6 \\ &\qquad \times 2039 \times 0 + 0.6 \times 20000 \times \sin(-6)\big) \\ &= 7618 \text{ N} \end{aligned}$$

$$F_{y2} = 454 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} = 1664 \text{ N}$$

$$\Delta F_{z2} = \frac{1847 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1297 \text{ N}$$

$$F_{z2i} = \frac{8454}{2} - 1363 = 2511 \text{ N}$$

$$F_{z2o} = \frac{8454}{2} + 1363 = 5106 \text{ N}$$

For Condition 4: The vehicle is accelerating at 0.6 m/s^2,

The forces on the axles (when traveling uphill) are

$$\begin{aligned} F_{z1} &= \frac{1}{3.0}\left(20{,}000 \times 1.8 \times \cos\left(-6\right) - 0.6 \times 397.2 - 0.6 \right.\\ &\qquad \left. \times\ 2039 \times 0.6 - 0.5 \times 20000 \times \sin\left(-6\right)\right) \\ &= 12{,}028 \text{ N} \end{aligned}$$

$$F_{y1} = 11{,}192 \times \frac{\left(27.78\right)^2}{2 \times 180 \times 9.81} = 2628 \text{ N}$$

$$\Delta F_{z1} = \frac{2445 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1644 \text{ N}$$

$$F_{z1i} = \frac{11{,}192}{2} - 1578 = 4370 \text{ N}$$

$$F_{z1o} = \frac{11{,}192}{2} + 1578 = 7658 \text{ N}$$

$$\begin{aligned} F_{z2} &= \frac{1}{3.0}\left(20{,}000 \times 1.2 \times \cos\left(-6\right) + 0.6 \times 397.2 + 0.6 \right.\\ &\qquad \left. \times\ 2039 \times 0.6 + 0.6 \times 20{,}000 \times \sin\left(-6\right)\right) \\ &= 7862 \text{ N} \end{aligned}$$

$$F_{y2} = 7862 \times \frac{\left(27.78\right)^2}{2 \times 180 \times 9.8} = 1717 \text{ N}$$

$$\Delta F_{z2} = \frac{1717 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1316 \text{ N}$$

$$F_{z2i} = \frac{7862}{2} - 1316 = 2614 \text{ N}$$

$$F_{z2o} = \frac{7862}{2} + 1316 = 5247 \text{ N}$$

Problem 9.10

Repeat Problem 9.9 when the vehicle is decelerating at 3.0 m/s^2 under braking effort.

Solution to Problem 9.10

For Condition 1: If the vehicle is traveling uphill and is decelerating at 3.0 m/s^2,

$$\begin{aligned} F_{z1} &= \frac{1}{3.0}\big(20{,}000 \times 1.8 \times \cos(6) - 0.6 \times 397.2 - 0.6 \\ &\qquad \times 2039 \times (-3) - 0.5 \times 20000 \times \sin(6)\big) \\ &= 12{,}660 \text{ N} \end{aligned}$$

$$F_{y1} = 12{,}660 \times \frac{(27.78)^2}{2 \times 180 \times 9.81} = 2766 \text{ N}$$

$$\Delta F_{z1} = \frac{2766 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1694 \text{ N}$$

$$F_{z1i} = \frac{12{,}660}{2} - 1694 = 4636 \text{ N}$$

$$F_{z1o} = \frac{12{,}660}{2} + 1694 = 8023 \text{ N}$$

$$\begin{aligned} F_{z2} &= \frac{1}{3.0}\big(20{,}000 \times 1.2 \times \cos(6) + 0.6 \times 397.2 + 0.6 \\ &\qquad \times 2039 \times (-3) + 0.6 \times 20000 \times \sin(6)\big) \\ &= 7230 \text{ N} \end{aligned}$$

$$F_{y2} = 7230 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} = 1580 \text{ N}$$

$$\Delta F_{z2} = \frac{1580 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1267 \text{ N}$$

$$F_{z2i} = \frac{7230}{2} - 1267 = 2348 \text{ N}$$

$$F_{z2o} = \frac{7230}{2} + 1267 = 4882 \text{ N}$$

For Condition 2: If the vehicle is traveling downhill and is decelerating at 3.0 m/s^2,

$$\begin{aligned} F_{z1} &= \frac{1}{3.0}\big(20{,}000 \times 1.8 \times \cos(-6) - 0.6 \times 397.2 - 0.6 \\ &\qquad \times 2039 \times (-3.0) - 0.5 \times 20000 \times \sin(-6)\big) \\ &= 13{,}496 \text{ N} \end{aligned}$$

$$F_{y1} = 13{,}496 \times \frac{(27.78)^2}{2 \times 180 \times 9.81} = 2949 \text{ N}$$

$$\Delta F_{z1} = \frac{2949 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1760 \text{ N}$$

$$F_{z1i} = \frac{13{,}496}{2} - 1760 = 4370 \text{ N}$$

$$F_{z1o} = \frac{13{,}496}{2} + 1760 = 8508 \text{ N}$$

$$\begin{aligned} F_{z2} &= \frac{1}{3.0}\big(20{,}000 \times 1.2 \times \cos(-6) + 0.6 \times 397.2 + 0.6 \\ &\qquad \times 2039 \times (-3) + 0.6 \times 20000 \times \sin(-6)\big) \\ &= 6394 \text{ N} \end{aligned}$$

$$F_{y2} = 6394 \times \frac{(27.78)^2}{2 \times 180 \times 9.8} = 1397 \text{ N}$$

$$\Delta F_{z2} = \frac{1397 \times 0.45 + 10{,}000 \times 0.08727}{1.25} = 1201 \text{ N}$$

$$F_{z2i} = \frac{6394}{2} - 1316 = 1996 \text{ N}$$

$$F_{z2o} = \frac{6394}{2} + 1316 = 4398 \text{ N}$$

Problem 9.11

Consider a vehicle with a 1500-kg mass when lightly loaded (driver only), with a CG at 1.0 m behind the front axle and at 0.45 m above the road. The mass is 2200 kg when loaded with a CG at 1.25 m behind the front axle and at 0.5 m above the road. The car is moving on a horizontal plane, and drag and rolling resistance are ignored. The car is going around a turning radius of 150 m at 120 km/hr. The wheelbase is 2.70 m, the wheel tread is 1.40 m, $K_\phi = 9000\ N \cdot m$, the roll center height is 0.4 m, and the roll angle is 5°. The vehicle is to achieve maximum possible braking force and front lock-up under the following conditions:

a. Vehicle is loaded, on a dry surface of $\mu = 0.9$

b. Vehicle is loaded, on a slippery surface of $\mu = 0.2$

c. Vehicle is lightly loaded, on a dry surface of $\mu = 0.9$

d. Vehicle is lightly loaded, on a slippery road of $\mu = 0.2$

Determine the value of the front brake force distribution factor K_f that is to be recommended, given that the brake system your company installs is conventional (no feedback control system). Determine the axle loads and the maximum braking force.

Solution to Problem 9.11

For each of the preceding cases, we could use the expression derived in Problem 9.7. With the availability of spreadsheets, we can find the optimal factor for each of the preceding cases.

Graphically, the a_F and a_R as a function of k_f are shown in Figure 9.11.

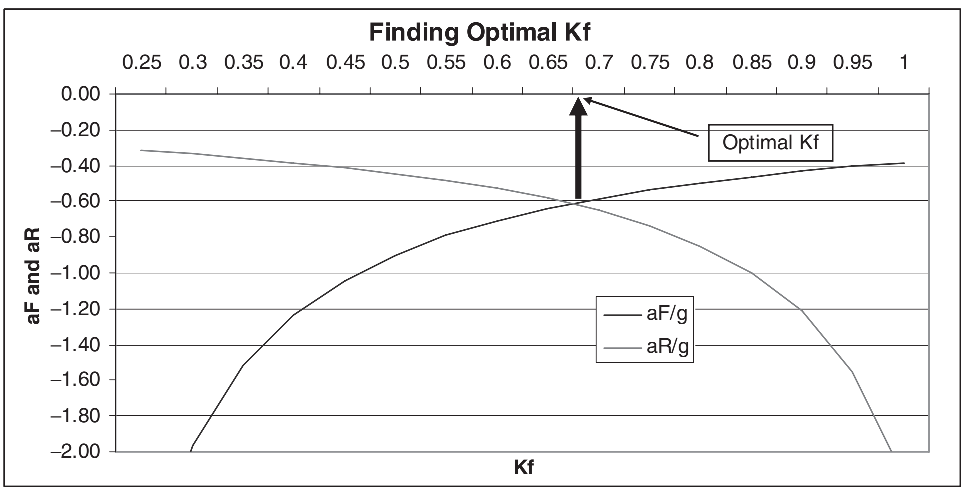

Figure 9.11 a_F and a_R as a function of k_f.

The results of Figure 9.11 are summarized as follows:

Case	c1	c2	h	μ	Mg	Kfmax
1	1.25	1.45	0.5	0.9	21582	0.6734
2	1.25	1.45	0.5	0.2	21582	0.5718
3	1	1.7	0.45	0.9	14715	0.7785
4	1	1.7	0.45	0.2	14715	0.6871

It is interesting to compare these results with those of Problem 7.11.

A distribution factor of 0.8 is recommended to ensure front lock-up. This is used to find the axle loads for each of the preceding conditions. The following data are generated:

Case	c1	c2	h	μ	Mg	Kf	aF/g	Fz1	Fz2	Fz	Fx1	Fx2	Fx	aR/g
1	1.25	1.45	0.5	0.9	21582	0.8	−0.496	13574	8008	21582	−12217	−3054	−15271	−0.847
2	1.25	1.45	0.5	0.2	21582	0.8	−0.096	11973	9609	21582	−2395	−599	−2993	−0.272
3	1	1.7	0.45	0.9	14715	0.8	−0.551	10616	4099	14715	−9554	−2389	−11943	−0.607
4	1	1.7	0.45	0.2	14715	0.8	−0.108	9530	5185	14715	−1906	−476	−2382	−0.189

Problem 9.12

For Problem 9.12, determine the maximum braking force if an anti-lock braking system (ABS) is installed instead of a conventional braking system.

Solution to Problem 9.12

In the case of ABS, the distribution factor is optimized for each case and for each wheel. This gives us the maximum friction force that is obtainable, which is the coefficient of friction multiplied by the normal Wight force. Thus, the braking force for each case is given as

Case	c1	c2	h	μ	Mg	Fx
1	1.25	1.45	0.5	0.9	21582	19424
2	1.25	1.45	0.5	0.2	21582	4316
3	1	1.7	0.45	0.9	14715	13244
4	1	1.7	0.45	0.2	14715	2943

Problem 9.13

Prove Eq. (9.26).

Solution to Problem 9.13

Forces normal to the road surface are summed to give the normal force

$$F_z = Mg\cos(\theta) + \frac{MV^2\sin(\theta)}{R} \tag{9.24}$$

The angle θ is the embankment angle and not the grade. The maximum friction force is

$$F_f = \frac{MV^2\cos(\theta)}{R} - Mg\sin(\theta) \tag{9.25}$$

Because $F_f = \mu F_z$ at the critical speed, Eq. 9.24 becomes

$$F_f = \mu\frac{MV_c^2\sin(\theta)}{R} + \mu Mg\cos(\theta)$$

Using the previous equation and Eq. 9.25, eliminating the mass M, collecting terms, and further manipulation, yields

$$\frac{V_c^{\,2}\left(1 - \mu \tan\left(\theta\right)\right)}{Rg} = \tan\left(\theta\right) + \mu$$

Isolating V_c yields

$$V_c = \frac{\sqrt{gR\left(\mu + \tan\theta\right)}}{\sqrt{1 - \mu \tan\theta}}$$

Problem 9.14

Derive an expression of critical speed similar to Eq. 9.26, taking into consideration braking and weight transfer.

Solution to Problem 9.14

During braking, weight transfers from the rear wheels to the front wheels. This will not impact the fundamental derivation of the equations.

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Chapter 10

Accident Reconstruction

Problem 10.1

A vehicle accelerates from a stop over a distance of 24.38 m (80 ft) in 7 s. Determine the following:

a. The speed of the vehicle after 9 s of acceleration from a stop if the acceleration is uniform over the entire 9 s

b. The distance traveled in the first 9 s

c. The speed of the vehicle after the first 4 s of acceleration

d. The time required to accelerate over the first 12.19 m (40 ft)

Solution to Problem 10.1

Here,

$v_i = 0$ ft/s

$d = 80$ ft (24.38 m)

$t = 7$ s

From Table 10.1,

$$a = \frac{2d - v_i t}{t^2} = \frac{2(80) - 2(0)(7)}{82} = \frac{160}{64} = 2.5 \text{ ft/s}^2 \ (0.76 \text{ m/s}^2)$$

a. Now,

$a = 2.5$ ft/s^2 (0.76 m/s^2)

$v_i = 0$ ft/s

$t = 9$ s

From Table 10.1,

$$v_e = v_i + at = 0 + (2.5)(9) = 22.5 \text{ ft/s } (6.86 \text{ m/s})$$

b. Here,

$v_i = 0$ ft/s

$a = 2.5$ ft/s^2

$t = 9$ s

From Table 10.1,

$$d = v_i t + \frac{1}{2}at^2 = 0(9) + \frac{1}{2}(2.5)(9^2) = 101.25 \text{ ft } (30.86 \text{ m})$$

c. For this case,

$v_i = 0$ ft/s

$a = 2.5$ ft/s^2

$t = 4$ s

From Table 10.1,

$$v_3 = v_1 + at = 0 + (2.5)(4) = 10 \text{ ft/s} = 6.80 \text{ mph } (10.94 \text{ km/hr})$$

d. In this case,

$d = 40$ ft

$v_i = 0$ ft/s

$a = 2.5$ ft/s^2

From Table 10.1,

$$t = \frac{v_e - v_i}{a}$$

and

$$\begin{aligned} v_e &= \sqrt{v_i^2 + 2ad} \\ &= \sqrt{0^2 + 2(2.5)(40)} \\ &= 14.14 \text{ ft/s } (4.31 \text{ m/s}) \end{aligned}$$

Therefore,

$$t = \frac{v_e - v_i}{a} = \frac{14.14 - 0}{2.5} = 5.66 \text{ s}$$

Problem 10.2

A vehicle skids for 45.72 m (150 ft) and hits a pedestrian. The vehicle continues to skid another 30.48 m (100 ft) before coming to a stop. The drag factor of the vehicle during skidding was 0.8. The pedestrian walked from the pavement edge northbound a distance of 4.57 m (15 ft) and then was struck by the vehicle. The pedestrian's working speed was 1.524 m/s (5 ft/s). Determine the following:

a. The initial speed of the vehicle

b. The velocity of the vehicle when the pedestrian was struck

c. The distance the vehicle is from its first contact position when the pedestrian first steps onto the pavement

Solution to Problem 10.2

a. For the initial speed of the vehicle,

$v_e = 0$

$a = fg = -0.8(32.2) = -25.76 \text{ ft/s}^2$ (-7.85 m/s^2)

$d = 150 + 100 = 250$ ft (76.2 m)

From Table G.1,

$$v_i = \sqrt{v_e^2 + 2ad} = \sqrt{0^2 - 2(-25.76)(250)} = 113.49 \text{ ft/s} = 77.20 \text{ mph (124.21 km/hr)}$$

b. For the vehicle velocity when the passenger was struck,

$v_e = 0$ ft/s

$a = -25.76 \text{ ft/s}^2$

$d = 100$ ft (30.48 m)

From Table G.1,

$$v_i = \sqrt{v_e^2 + 2ad} = \sqrt{0^2 - 2(-25.76)(100)} = 71.78 \text{ ft/s} = 48.83 \text{ mph} \ \ (78.57 \text{ km/hr})$$

c. The time the pedestrian walks to the collision point is the same time that the vehicle takes to move toward the collision point. Hence, we first calculate the time for the pedestrian to walk to the collision point for a constant speed of 5 ft/s and a distance of 15 ft, where

$d = 15$ ft

$v = 5$ ft/s

From Table G.1,

$$t = d/v = 15/5 = 3\ \text{s}$$

Now, we determine where the vehicle was 3 s before the collision. We will compute the time it took for the vehicle to skid 150 ft before the collision. Now,

$v_i = 113.49$ ft/s

$v_e = 71.78$ ft/s

$a = -25.76$ ft/s^2 (-7.85 m/s^2)

From Table 10.1,

$$t = \frac{v_e - v_i}{a} = \frac{71.78 - 113.49}{-25.76} = 1.62\ \text{s}$$

Hence, of the 3 s that the pedestrian takes to walk toward the collision, the vehicle is skidding for 1.62 s. Hence, for the remaining time (3 − 1.62 = 1.38 s), the vehicle travels at the speed calculated at the beginning of the skid (113.49 ft/s). Hence, the distance traveled for 1.38 s is

$$d = vt = (113.49)(1.38) = 156.62\ \text{ft}\ (47.74\ \text{m})$$

The distance the vehicle is from its first contact position, when the pedestrian first steps onto the pavement, is given by the sum of 156.62 ft plus 150 ft (i.e., the distance skidded before impact).

Hence, the total distance is

$$156.62 + 150 = 306.32\ \text{ft}\ (93.46\ \text{m})$$

Problem 10.3

The height of the center of mass of a vehicle is 0.64 m (2.1 ft) and is located 1.31 m (4.3 ft) behind the front axle. The wheelbase of the car is 2.74 m (9.5 ft). Assume that no braking occurs on the front axle, resulting in a drag factor due to rolling resistance of 0.01. Also, the rear wheels are locked by brakes and slide on a pavement with a coefficient of friction of 0.70. Determine the drag factor for the whole vehicle.

Solution to Problem 10.3

The drag for the whole vehicle can be obtained from Eq. G.5:

$$f_R = \frac{f_f - x_f(f_f - f_r)}{1 - z(f_f - f_r)} \tag{10.1}$$

Here,

$$
\begin{aligned}
z &= \frac{2.1}{9.5} = 0.221 \\
x_f &= \frac{4.3}{9.5} = 0.453 \\
f_f &= 0.1 \\
f_r &= 0.70
\end{aligned}
$$

When the above values are substituted in Eq. 10.1, we have

$$f_R = \frac{0.01 - 0.453(0.01 - 0.70)}{1 - 0.221(0.01 - 0.70)} = \frac{0.323}{1.152} = 0.28$$

Problem 10.4

The measured values of chord and middle ordinate from a test of a sudden maneuver of a passenger road vehicle are 14.63 m (48 ft) and 0.46 m (1.5 ft). Assume an average frictional drag coefficient of 0.75. The test site has a zero grade and zero super elevation. Determine the following:

a. The speed over the measured arc of the yaw marks

b. The frictional drag coefficient that gives a speed of 96.54 km/hr (60 mph)

Solution to Problem 10.4

a. From Eq. 10.44, the radius of the circular arc is

$$R = \frac{c^2}{8M} + \frac{M}{2} = \frac{48^2}{8(1.5)} + \frac{1.5}{2} = 192.75 \text{ ft } (58.75 \text{ m})$$

The speed of the vehicle is given by Eq. 10.49 as

$$V_{cr} = \sqrt{15Rf} = \sqrt{15(192.75)(0.75)} = 46.57 \text{ mph } (74.93 \text{ km/hr})$$

b. From Eq. 10.49,

$$f = \frac{V_{cr}^2}{15R} = \frac{48^2}{15(192.75)} = 0.797$$

Problem 10.5

Determine the highest speed of a road vehicle that can travel with a circular path on a level unbanked road with a 335.28-m (1100-ft) radius under icy conditions, with $f = 0.08$.

Solution to Problem 10.5

From Eq. 10.50 with $e = \tan\gamma - 0$,

$$V_{cr} = \sqrt{gRf} = \sqrt{32.2(1100)(0.08)} = 53.23 \text{ ft/s } (85.65 \text{ km/hr})$$

Problem 10.6

A four-wheeled vehicle had brakes on only its front wheels. When the brakes were applied, this caused the vehicle to skid to a stop.

a. Determine the speed of the vehicle from the following information:

   Skid distance = 33.53 m (110 ft) on level surface

   Drag factor = 0.80 (level surface)

   Weight on front wheels during braking = 65% of total vehicle weight

b. If the vehicle is traveling on a downhill grade of 5% and the drag factor is 0.82, calculate the speed of the vehicle.

Solution to Problem 10.6

a. To determine the speed of the vehicle (English),

$$S = \sqrt{30\,D\,f\,n}$$

   To determine the speed of the vehicle (S.I. units),

$$\begin{aligned} S &= \sqrt{254\,D\,f\,n} \\ &= \sqrt{30(110)(0.80)(0.65)} \\ &= \sqrt{254(33.53)(0.80)(0.65)} \\ &= 41.43 \text{ mph } (66.55 \text{ km/hr}) \end{aligned}$$

b. To calculate the downhill speed (English),

$$S = \sqrt{30\,D(\mu n \pm m)}$$

   To calculate the downhill speed (S.I. units),

$$\begin{aligned} S &= \sqrt{254\,D(\mu n \pm m)} \\ &= \sqrt{30(110)\left[(0.80)(0.65) - 0.05\right]} \\ &= \sqrt{254(33.53)\left[(0.80)(0.65) - 0.05\right]} \\ &= 39.38 \text{ mph } (63.27 \text{ km/hr}) \end{aligned}$$

Problem 10.7

A vehicle with braking on all four wheels was towing a trailer that was not equipped with brakes. In an attempt to avoid a collision with another vehicle, the driver applied the brakes of the vehicle. Determine the speed of the vehicle and trailer at the beginning of the skid marks from the vehicle by using the following information:

Vehicle weight = 1587.60 kg (3500 lb)

Trailer weight = 680.40 kg (1500 lb)

Skid distance = 19.81 m (65 ft)

Drag factor = 0.65

Solution to Problem 10.7

In English units,

$$S = \sqrt{30\,D\,f\,n}$$

In S.I. units,

$$\begin{aligned} S &= \sqrt{254\,D\,f\,n} \\ &= \sqrt{30(65)(0.65)(0.7)} \\ &= \sqrt{254(19.81)(0.65)(0.7)} \\ &= 29.79 \text{ mph } (47.85 \text{ km/hr}) \end{aligned}$$

where $n = \dfrac{3500}{5000} = 0.7$.

Problem 10.8

The operator of a motorcycle applied only the rear wheel brake and caused the motorcycle to skid to a stop. Determine the speed of the motorcycle at the beginning of its skid mark by using the following information:

Weight of motorcycle and rider = 358.56 kg (850 lb)

Weight on front wheel of motorcycle = 158.76 kg (350 lb)

Weight on rear wheel of motorcycle = 226.80 kg (500 lb)

20% forward shift during braking = 49.90 kg (110 lb)

Total weight on rear wheel during braking action = 50 – 110 = 176.90 kg (390 lb)

Skid distance = 19.81 m (65 ft)

Drag factor = 0.65

Solution to Problem 10.8

$$n = \frac{390}{850} = \frac{176.90}{385.56} = 0.46$$

In English,

$$S = \sqrt{30\,D\,f\,n}$$

In S.I. units,

$$\begin{aligned} S &= \sqrt{254\,D\,f\,n} \\ &= \sqrt{30(65)(0.65)(0.46)} \\ &= \sqrt{254(19.81)(0.65)(0.46)} \\ &= 24.15 \text{ mph } (38.78 \text{ km/hr}) \end{aligned}$$

Problem 10.9

The driver of a vehicle applied the brakes of the vehicle to avoid an accident and caused the vehicle to skid to a stop. The left wheels of the vehicle were on an asphalt surface, and the right wheels were on a gravel surface during the entire skid. Determine the speed of the vehicle at the beginning of the skid marks by using the following information:

Skid distance = 30.48 m (100 ft)

Drag factor for asphalt surface = 0.70

Drag factor for gravel surface = 0.45

Solution to Problem 10.9

In English units,

$$S = \sqrt{15(f_1 + f_2)D}$$

In S.I. units,

$$\begin{aligned} S &= \sqrt{127(f_1 + f_2)D} \\ &= \sqrt{15(0.70 + 0.45)(100)} \\ &= \sqrt{127(0.7 + 0.45)(30.48)} \\ &= 41.53 \text{ mph } (66.72 \text{ km/hr}) \end{aligned}$$

Problem 10.10

A vehicle traveling on a curve sideslipped, leaving a yaw mark 100.58 m (330 ft) in length. The first one-third (33.53 m; 110 ft) of the yaw mark was used, and a chord of 21.34 m (70 ft) and a middle ordinate of 1.07 m (3.5 ft) were measured. The drag factor is known to be 0.80. The curve had a super elevation of +06%. Determine the following:

a. The radius of the yaw mark

b. The critical vehicle curve speed from the yaw mark

Solution to Problem 10.10

a. The radius of the yaw mark is

$$
\begin{aligned}
R &= \frac{C^2}{8M} + \frac{M}{2} \\
&= \frac{70^2}{8(3.5)} + \frac{3.5}{2} \\
&= \frac{21.34^2}{8(1.07)} + \frac{1.07}{2} \\
&= 176.75 \text{ ft } (53.74 \text{ m})
\end{aligned}
$$

b. The critical vehicle curve speed (English units) is

$$S = 3.86\sqrt{R(f \pm e)}$$

The critical vehicle curve speed (S.I. units) is

$$
\begin{aligned}
S &= 11.27\sqrt{R(f \pm e)} \\
&= 3.86\sqrt{176.75(0.80 + 0.06)} \\
&= 11.27\sqrt{53.75(0.80 + 0.06)} \\
&= 47.59 \text{ mph } (76.62 \text{ km/hr})
\end{aligned}
$$

Problem 10.11

A road vehicle was traveling on a level asphalt road when it struck a utility pole head-on. At the time of impact, a carton in the roof rack was projected forward and landed on the road in front of the vehicle. The base of the pole was assumed to be at road level. The height of the carton at the point of takeoff was 1.83 m (6 ft). The horizontal distance the carton was projected at was 9.14 m (30 ft). All measurements were taken from center of mass to center of mass. Determine the speed of the vehicle.

Solution to Problem 10.11

In English units,

$$S = \frac{2.73D}{\sqrt{h}}$$

In S.I. units,

$$S = \frac{7.97D}{\sqrt{h}} = \frac{2.73(30)}{\sqrt{6}} = \frac{7.97(9.14)}{\sqrt{1.83}} = 33.44 \text{ mph } (53.85 \text{ km/hr})$$

Problem 10.12

Upon striking an unforeseen object with its front end, a vehicle vaulted through the air and landed on its roof on a lawn. The takeoff angle was assumed as 45°, and the center of mass at landing was at the same level as at takeoff. The distance between the two center-of-mass positions was 15.24 m (50 ft). Determine the speed at takeoff.

Solution to Problem 10.12

In English units,

$$S = \sqrt{15\,D}$$

In S.I. units,

$$S = \sqrt{127\,D} = \sqrt{15(50)} = \sqrt{127(15.24)} = 27.39 \text{ mph (43.99 km/hr)}$$

Problem 10.13

A vehicle was traveling on a highway, and upon approaching a dip in the road, the vehicle momentarily became airborne and landed on the roadway at some distance. Determine the speed of the vehicle at the point of takeoff with the following information:

Horizontal distance = 27.43 m (90 ft)

Vertical distance (lower) = 0.91 m (3 ft)

Angle of departure = 5°

Solution to Problem 10.13

In English units,

$$S = \frac{2.73\,D}{\sqrt{D\sin\theta\cos\theta + h\cos^2\theta}}$$

In S.I. units,

$$\begin{aligned} S &= \frac{7.97\,D}{\sqrt{D\sin\theta\cos\theta + h\cos^2\theta}} \\ &= \frac{2.73(90)}{\sqrt{90(0.087)(0.996) + 3(0.996)^2}} \\ &= \frac{7.97(27.43)}{\sqrt{27.43(0.087)(0.996) + (0.91)(0.996)^2}} \\ &= 74.86 \text{ mph (120.72 km/hr)} \end{aligned}$$

where

$\cos\theta = \cos 5^\circ = 0.996$

$\sin\theta = \sin 5^\circ = 0.087$

Problem 10.14

A four-wheeled vehicle entered into a T intersection, braked, skidded on an asphalt surface, and then went into a fall from the highway. Determine the following:

a. The speed from skid marks

b. The fall speed

c. The combined speed

The following are given:

Skid distance on asphalt surface = 24.38 m (80 ft)

Drag factor = 0.65

Grade at takeoff point = 0°

Horizontal distance of fall = 27.43 m (90 ft)

Vertical distance of fall = 10.67 m (35 ft)

Solution to Problem 10.14

a. The speed from the skid marks (English units) is

$$S_1 = \sqrt{30\,D\,f}$$

The speed from the skid marks (S.I. units) is

$$\begin{aligned} S_1 &= \sqrt{254\,D\,f} \\ &= \sqrt{30(80)(0.65)} \\ &= \sqrt{254(24.38)(0.65)} \\ &= 39.49 \text{ mph } (63.44 \text{ km/hr}) \end{aligned}$$

b. The fall speed (English units) is

$$S_2 = \frac{2.73\,D}{\sqrt{h}}$$

The fall speed (S.I. units) is

$$\begin{aligned} S_2 &= \frac{797\,D}{\sqrt{h}} \\ &= \frac{2.73(80)}{\sqrt{35}} \\ &= \frac{7.97(24.38)}{\sqrt{10.67}} \\ &= 36.92 \text{ mph } (59.49 \text{ km/hr}) \end{aligned}$$

c. The combined speed (English units) is

$$S_c = \sqrt{S_1^2 + S_2^2}$$

The combined speed (S.I. units) is

$$\begin{aligned} S_c &= \sqrt{S_1^2 + S_2^2} \\ &= \sqrt{39.49^2 + 36.92^2} \\ &= \sqrt{63.44^2 + 59.49^2} \\ &= 54.06 \text{ mph } (86.97 \text{ km/hr}) \end{aligned}$$

Problem 10.15

Two vehicles, Vehicles 1 and 2, were traveling in side-by-side lines at 40.23 km/hr (25 mph). After passing through an intersection traffic light, Vehicle 2 accelerated sharply, and at a distance of 106.68 m (350 ft) from the traffic light, Vehicle 2 struck a pedestrian who was in a crosswalk. Determine the following:

a. The speed of Vehicle 2 when it struck the pedestrian

b. The time it took for Vehicle 1 to accelerate from its initial and final speeds

Assume that the maximum acceleration rate is 3.66 m/s^2 (12 ft/s^2).

Solution to Problem 10.15

a. In English units,

$$V = S(1.466)$$

$$V = S(0.278) = 25(1.466) = 40.23(0.278) = 36.65 \text{ ft/s} = 11.18 \text{ m/s}$$

The speed of Vehicle 2 when it struck the pedestrian is

$$\begin{aligned} V_f &= \sqrt{V_0^2 + 2\,a\,D} \\ &= \sqrt{36.65^2 + 2(12)(350)} \\ &= \sqrt{11.18^2 + 2(3.66)(106.68)} \\ &= 98.75 \text{ ft/s } (30.09 \text{ m/s}) \end{aligned}$$

or

$$V_f = \frac{98.71}{1.466} = 67.33 \text{ mpg } (108.27 \text{ km/hr})$$

b. The time it took Vehicle 1 to travel 350 ft (106.68 m) under acceleration (English units) is given by

$$t = \frac{V_f - V_0}{a} = \frac{98.71 - 36.65}{12} = \frac{30.09 - 11.18}{3.66} = 5.17 \text{ s}$$

Problem 10.16

A vehicle traveling at 80.45 km/hr (50 mph) skidded a distance of 30.48 m (100 ft) to a stop on a roadway with a drag factor of 0.80. Determine the time it took the vehicle to skid the 100-ft distance.

Solution to Problem 10.16

The time it took the vehicle to skid 30.48 m (100 ft) (English units) is given by

$$t = 0.45\sqrt{D/f} = 0.249\sqrt{100/0.80} = 0.45\sqrt{30.48/0.80} = 2.78 \text{ s}$$

Problem 10.17

Determine the impact speeds of Vehicles 1 and 2 from the following:

Weight of Vehicle 1 = 2041.20 kg (4500 lb)

Weight of Vehicle 2 = 1451.52 kg (3200 lb)

Approach angle of Vehicle 1 = 0°

Approach angle of Vehicle 2 = 150°

Departure angle of Vehicle 1 = 30°

Departure angle of Vehicle 2 = 12°

Post-impact speed of Vehicle 1 = 46.66 km/hr (29 mph)

Post-impact speed of Vehicle 2 = 43.44 km/hr (27 mph)

Figure 10.17 shows the free body diagram for this scenario.

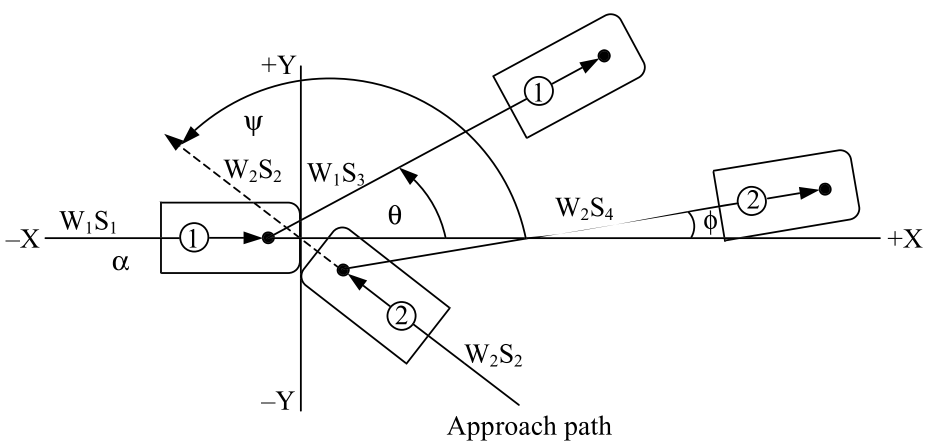

Figure 10.17 Free body diagram.

Solution to Problem 10.17

In English units,

Vehicle 1: W_1 = 2041.20 kg (4500 lb)

Vehicle 2: W_2 = 1451.52 kg (3200 lb)

The approach angles are

Vehicle 1: $\alpha = 0°$ $\sin = 0$ $\cos = 1$

Vehicle 2: $\psi = 150°$ $\sin = 0.5$ $\cos = -0.866$

The departure angles are

Vehicle 1: $\theta = 30°$ $\sin = 0.5$ $\cos = 0.866$

Vehicle 2: $\phi = 12°$ $\sin = 0.2079$ $\cos = 0.978$

The post-impact speeds are

Vehicle 1: $S_3 = 46.66$ km/hr (29 mph)

Vehicle 2: $S_4 = 43.44$ km/hr (27 mph)

The impact speed of Vehicle 2 (in English units) is given by

$$
\begin{aligned}
S_2 &= \frac{W_1 S_3 \sin\theta}{W_2 \sin\psi} + \frac{S_4 \sin\phi}{\sin\psi} \\
&= \frac{4500(29)(0.5)}{3200(0.5)} + \frac{27(0.2079)}{0.5} \\
&= \frac{(2041.20)(46.66)(0.5)}{1451.52(0.5)} + \frac{43.44(0.2079)}{0.5} \\
&= 83.68 \text{ km/hr (52 mph)}
\end{aligned}
$$

The impact speed of Vehicle 1 (in English units) is

$$
\begin{aligned}
S_1 &= S_3 \cos\theta + \frac{W_2 S_4 \cos\phi}{W_1} - \frac{W_2 S_2 \cos\psi}{W_2} \\
&= 29(0.866) + \frac{3200(27)(0.978)}{4500} - \frac{3200(52)(-0.866)}{4500} \\
&= 75.91 \text{ mph}
\end{aligned}
$$

The impact speed of Vehicle 1 (in S.I. units) is

$$
\begin{aligned}
S_1 &= S_3 \cos\theta + \frac{W_2 S_4 \cos\phi}{W_1} - \frac{W_2 S_2 \cos\psi}{W_2} \\
&= (46.66)(0.866) + \frac{1451.52(43.44)(0.978)}{2041.20} - \frac{1451.52(83.68)(-0.866)}{2041.20} \\
&= 122.15 \text{ km/hr}
\end{aligned}
$$

Problem 10.18

Determine the impact speeds of Vehicles 1 and 2 from the following information:

Weight of Vehicle 1 = 1587.60 kg (3500 lb)

Weight of Vehicle 2 = 1134 kg (2500 lb)

Approach angle of Vehicle 1 = 0°

Approach angle of Vehicle 2 = 250°

Departure angle of Vehicle 1 = 300°

Departure angle of Vehicle 2 = 330°

Post-impact speed of Vehicle 1 = 51.49 km/hr (32 mph)

Post-impact speed of Vehicle 2 = 43.44 km/hr (27 mph)

Solution to Problem 10.18

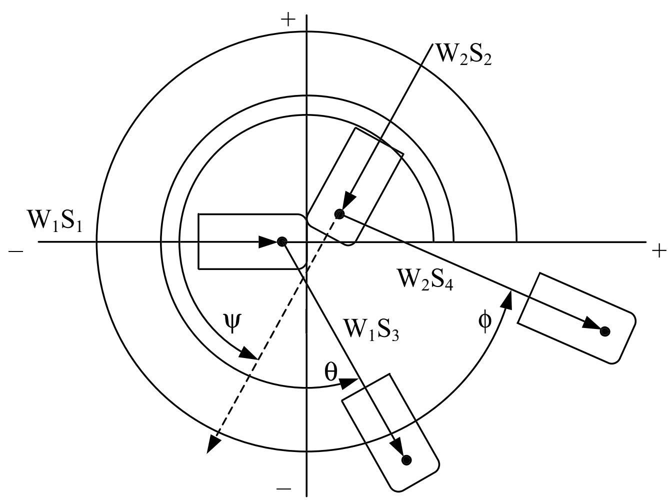

Figure 10.18 Free body diagram.

The vehicle weights are

Vehicle 1: $W_1 = 1587.60$ kg (3500 lb)

Vehicle 2: $W_2 = 1134$ kg (2500 lb)

The approach angles are

Vehicle 1:	$\alpha = 0°$	sin = 0	cos = 1.0
Vehicle 2:	$\psi = 250°$	sin = –0.9396	cos = –0.3420

The departure angles are

Vehicle 1:	$\theta = 300°$	sin = –0.8660	cos = 0.5
Vehicle 2:	$\phi = 330°$	sin = –0.5	cos = 0.8660

The post-impact speeds are

Vehicle 1: S3 = 51.49 km/hr (32 mph)

Vehicle 2: S4 = 43.44 km/hr (27 mph)

The impact speed of Vehicle 2 (English units) is

$$S_2 = \frac{W_1 S_3 \sin\theta}{W_2 \sin\psi} + \frac{S_4 \sin\phi}{\sin\psi}$$

The impact speed of Vehicle 2 (S.I. units) is

$$\begin{aligned} S_2 &= \frac{W_1 S_3 \sin\theta}{W_2 \sin\psi} + \frac{S_4 \sin\phi}{\sin\psi} \\ &= \frac{3500(32)(-0.866)}{2500(-0.9396)} + \frac{27(-0.5)}{(-0.9396)} \\ &= \frac{1587.60(51.49)(-0.866)}{1134(-0.9396)} + \frac{43.44(-0.5)}{(-0.9396)} \\ &= 52 \text{ mph } (89.56 \text{ km/hr}) \end{aligned}$$

The impact speed of Vehicle 1 (English units) is

$$\begin{aligned} S_1 &= S_3 \cos\theta + \frac{W_2 S_4 \cos\phi}{W_1} - \frac{W_2 S_2 \cos\psi}{W_2} \\ &= 32(0.5) + \frac{2500(27)(0.866)}{3500} - \frac{2500(55.66)(-0.342)}{3500} \\ &= 46.30 \text{ mph} \end{aligned}$$

The impact speed of Vehicle 1 (S.I. units) is

$$\begin{aligned} S_1 &= S_3 \cos\theta + \frac{W_2 S_4 \cos\phi}{W_1} - \frac{W_2 S_2 \cos\psi}{W_2} \\ &= 51.49(0.5) + \frac{(1134)(43.44)(0.866)}{1587.60} - \frac{1134(89.56)(-0.342)}{1587.60} \\ &= 74.49 \text{ km/hr} \end{aligned}$$

Problem 10.19

Determine the impact speeds of Vehicles 1 and 2 from the following information:

Weight of Vehicle 1 = 1360.80 kg (3000 lb)

Weight of Vehicle 2 = 2358.72 kg (5200 lb)

Approach angle of Vehicle 1 = 0°

Approach angle of Vehicle 2 = 90°

Departure angle of Vehicle 1 = 77°

Departure angle of Vehicle 2 = 40°

Post-impact speed of Vehicle 1 = 56.32 km/hr (35 mph)

Post-impact speed of Vehicle 2 = 61.14 km/hr (38 mph)

Solution to Problem 10.19

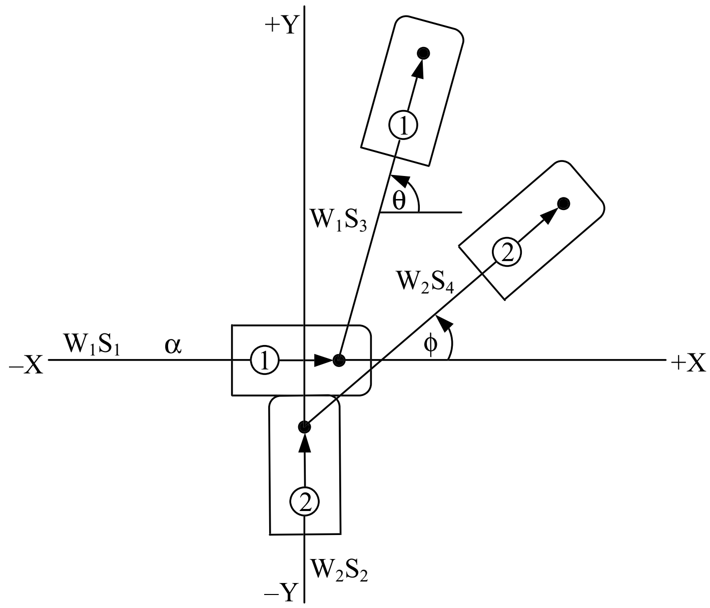

Figure 10.19 Free body diagram.

The vehicle weights are

Vehicle 1: W1 = 1360.80 kg (3000 lb)

Vehicle 2: W2 = 2358.72 kg (5200 lb)

The approach angles are

Vehicle 1:	$\alpha = 0°$	sin = 0	cos = 1.0
Vehicle 2:	$\psi = 90°$	sin = 1.0	cos = 0

The departure angles are

Vehicle 1:	$\theta = 77°$	sin = 0.9743	cos = 0.2249
Vehicle 2:	$\phi = 40°$	sin = 0.6427	cos = 0.7660

The post-impact speeds are

Vehicle 1: $S_3 = 56.32$ km/hr (35 mph)

Vehicle 2: $S_4 = 61.14$ km/hr (38 mph)

The impact speed of Vehicle 2 (90° collision) is

$$S_2 = \frac{W_1 S_3 \sin\theta}{W_2} + S_4 \sin\phi$$

$$= \frac{3000(35)(0.9743)}{5200} + 38(0.6427)$$

$$= \frac{1360.80(56.32)(0.9743)}{2358.72} + 61.14(0.6427)$$

$$= 70.94 \text{ km/hr } (44.09 \text{ mph})$$

The impact speed of Vehicle 1 is

$$\begin{aligned} S_1 &= S_3 \cos\theta + \frac{W_2 S_4 \cos\phi}{W_1} \\ &= 35(0.2249) + \frac{5200(38)(0.766)}{3000} \\ &= 58.33 \text{ mph} \end{aligned}$$

or

$$\begin{aligned} S_1 &= S_3 \cos\theta + \frac{W_2 S_4 \cos\phi}{W_1} \\ &= 56.32(0.2249) + \frac{2358.72(61.14)(0.766)}{1360.80} \\ &= 73.85 \text{ km/hr} \end{aligned}$$

Problem 10.20

Vehicles 1 and 2 were traveling along a roadway. Vehicle 1 overtook Vehicle 2 and struck the rear of that vehicle while both vehicles were moving. Determine the following:

a. The departure speed of Vehicle 1 based on its skid distance

b. The departure speed of Vehicle 2 based on its skid distance

c. The impact speed of Vehicle 1 in an in-line rear-end collision

Given are the following:

Weight of Vehicle 1 = 1769.04 kg (3900 lb)

Weight of Vehicle 2 = 2041.20 kg (4500 lb)

Post-impact skid distance of Vehicle 1 = 19.75 m (32 ft)

Post-impact skid distance of Vehicle 2 = 12.80 m (42 ft)

Drag factor = 0.80

Braking efficiency for both vehicles = 100%

Assume that the speed of Vehicle 2 (S_2) at the time of impact = 20 mph (32 km/hr)

Solution to Problem 10.20

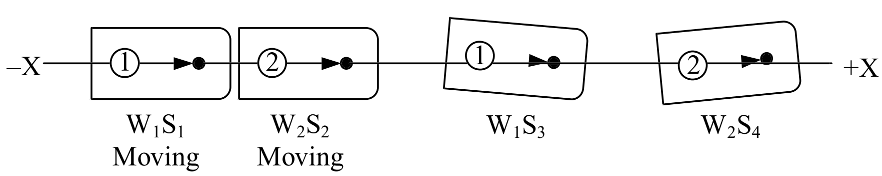

Figure 10.20 Free body diagram.

The vehicle weights are

Vehicle 1: $W_1 = 1769.04$ kg (3900 lb)

Vehicle 2: $W_2 = 2041.20$ kg (4500 lb)

The post-impact skid distances are

Vehicle 1: $D_3 = 9.75$ m (32 ft)

Vehicle 2: $D_4 = 12.80$ m (42 ft)

Drag factor = 0.80

Braking efficiency = 100% for both vehicles

a. The departure speed (S_3) of Vehicle 1 (English units) based on its skid distance (D_3) is

$$S_3 = \sqrt{30\, D_3\, f}$$

The departure speed (S_3) of Vehicle 1 (S.I. units) based on its skid distance (D_3) is

$$\begin{aligned} S_3 &= \sqrt{254\, D_3\, f} \\ &= \sqrt{30(32)(0.80)} \\ &= \sqrt{254(9.75)(0.80)} \\ &= 27.71 \text{ mph } (44.51 \text{ km/hr}) \end{aligned}$$

b. The departure speed (S_4) of Vehicle 2 (English units) based on its skid distance (D_4) is

$$S_4 = \sqrt{30\, D_4\, f}$$

The departure speed (S_4) of Vehicle 2 (S.I. units) based on its skid distance (D_4) is

$$\begin{aligned} S_4 &= \sqrt{254\, D_4\, f} \\ &= \sqrt{30(42)(0.80)} \\ &= \sqrt{254(12.80)(0.8)} \\ &= 31.75 \text{ mph } (49.38 \text{ km/hr}) \end{aligned}$$

c. The impact speed (S_1) of Vehicle 1 in an in-line rear-end collision is

$$\begin{aligned} S_1 &= S_3 + \frac{W_2}{W_1} S_4 - \frac{W_2}{W_1} S_2 \\ &= 27.71 + \frac{4500}{3900}(31.75) - \frac{4500}{3900}(20) \\ &= 44.51 + \frac{2041.20}{1769.04}(49.38) - \frac{2041.20}{1769.04}(32) \\ &= 41.27 \text{ mph } (66.40 \text{ km/hr}) \end{aligned}$$

Problem 10.21

Vehicle 1 was traveling along a roadway and collided with the rear end of Vehicle 2, which was parked on the roadway. Determine the following:

a. The pre-impact (lead-in) speed loss of Vehicle 1 based on its pre-impact skid distance

b. The post-impact speed loss of vehicle 1 based on its post-impact skid distance

c. The post-impact speed of Vehicle 2 based on its post-impact skid distance

d. The impact speed of Vehicle 1

e. The speed of Vehicle 1 at the beginning of its lead-in (pre-impact) skid mark

Given are the following:

Weight of Vehicle 1 = 1587.60 kg (3500 lb)

Weight of Vehicle 2 = 907.20 kg (2000 lb)

Pre-impact skid distance of Vehicle 1 = 9.45 m (31 ft)

Pre-impact skid distance of Vehicle 2 = 6.71 m (22 ft)

Drag factor = 0.85

Solution to Problem 10.21

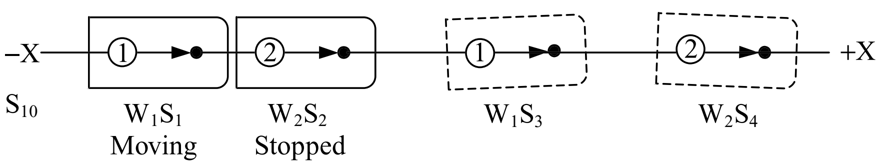

Figure 10.21 Free body diagram.

The vehicle weights are

Vehicle 1: W_1 = 1587.60 kg (3500 lb)

Vehicle 2: W_2 = 907.20 kg (2000 lb)

The pre-impact skid distances are

Vehicle 1: D_1 = 9.45 m (31 ft)

Vehicle 2: D_2 = 6.71 m (22 ft)

The post-impact skid distances are

Vehicle 1: D_3 = 4.88 m (16 ft)

Vehicle 2: D_4 = 6.71 m (22 ft)

Drag factor = 0.85

a. The pre-impact (lead-in) speed loss (S_{1p}) of Vehicle 1 (English units) based on pre-impact skid distance (D_1) is

$$S_{1p} = \sqrt{30\, D_1\, f}$$

The pre-impact (lead-in) speed loss (S_{1p}) of Vehicle 1 (S.I. units) based on pre-impact skid distance (D_1) is

$$\begin{aligned} S_{1p} &= \sqrt{254\, D_1\, f} \\ &= \sqrt{30(31)(0.85)} \\ &= \sqrt{254(9.45)(0.85)} \\ &= 28.12 \text{ mph } (45.17 \text{ km/hr}) \end{aligned}$$

b. The post-impact speed (S_3) of Vehicle 2 (English units) based on its post-impact skid distance (D_3) is

$$S_3 = \sqrt{30\, D_3\, f}$$

The post-impact speed (S_3) of Vehicle 2 (S.I. units) based on its post-impact skid distance (D_3) is

$$\begin{aligned} S_3 &= \sqrt{254\, D_3\, f} \\ &= \sqrt{30(16)(0.85)} \\ &= \sqrt{254(4.88)(0.85)} \\ &= 20.20 \text{ mph } (32.50 \text{ km/hr}) \end{aligned}$$

c. The post-impact speed (S_4) of Vehicle 2 (English units) based on its post-impact skid distance (D_4) is

$$S_4 = \sqrt{30\, D_4\, f}$$

The post-impact speed (S_4) of Vehicle 2 (S.I. units) based on its post-impact skid distance (D_4) is

$$\begin{aligned} S_4 &= \sqrt{254\, D_4\, f} \\ &= \sqrt{30(22)(0.85)} \\ &= \sqrt{254(6.71)(0.85)} \\ &= 23.69 \text{ mph } (38.11 \text{ km/hr}) \end{aligned}$$

d. The impact speed (S_1) of Vehicle 1 is

$$\begin{aligned} S_1 &= S_3 + \frac{W_2}{W_1} S_4 \\ &= 20.20 + \frac{2000}{3500}(23.69) \\ &= 32.50 + \frac{907.20}{1587.60}(38.11) \\ &= 33.74 \text{ mph } (54.28 \text{ km/hr}) \end{aligned}$$

Problem 10.22

Vehicles 1 and 2 traveling on a highway collided head-on. The drivers of both vehicles applied their brakes, and the vehicles skidded to the point of impact. The vehicles had very little rotation after impact. Vehicle 1 continued a short distance in its original direction of travel, and Vehicle 2 was forced back a short distance along its original path of travel. The following information is known:

Weight of Vehicle 1 = 1587.60 kg (3500 lb)

Weight of Vehicle 2 = 907.20 kg (2000 lb)

Pre-impact skid distance of Vehicle 1 = 9.75 m (32 ft)

Pre-impact skid distance of Vehicle 2 = 14.33 m (47 ft)

Post-impact skid distance of Vehicle 1 = 4.88 m (16 ft)

Post-impact skid distance of Vehicle 2 = 4.88 m (16 ft)

Drag factor = 0.85

Determine the following:

a. Pre-impact (lead-in) speed loss based on pre-impact skid distance of Vehicle 1

b. Pre-impact (lead-in) speed loss based on pre-impact skid distance of Vehicle 2

c. Post-impact speed of Vehicle 1 based on its post-impact skid distance

d. Post-impact speed of Vehicle 2

e. Impact speed of Vehicle 1

f. Impact speed of Vehicle 2

g. Speed of Vehicle 1 at the beginning of its lead-in (pre-impact) skid marks

h. Speed of Vehicle 2 at the beginning of its lead-in (pre-impact) skid marks

Solution to Problem 10.22

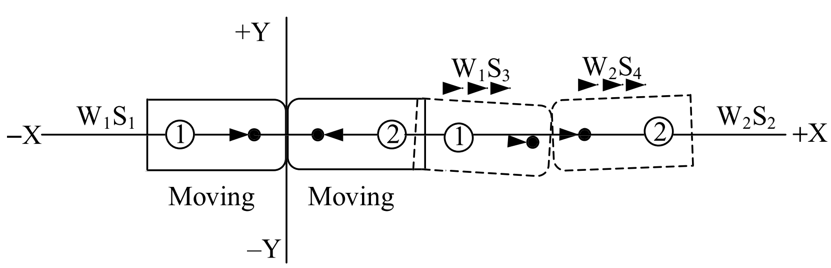

Figure 10.22 Free body diagram.

The vehicle weights are

Vehicle 1: W_1 = 1587.60 kg (3500 lb)

Vehicle 2: W_2 = 907.20 kg (2000 lb)

The pre-impact skid distances are

Vehicle 1: $D_{1p} = 9.75$ m (32 ft)

Vehicle 2: $D_1 = 14.33$ m (47 ft)

The post-impact skid distances are

Vehicle 1: $D_3 = 4.88$ m (16 ft)

Vehicle 2: $D_4 = 4.88$ m (16 ft)

Drag factor = 0.85

a. Pre-impact (lead-in) speed loss (S_{1p}) (English units) based on pre-impact skid distance (D_1):

$$S_{1p} = \sqrt{30\,D_1\,f}$$

Pre-impact (lead-in) speed loss (S_{1p}) (S.I. units) based on pre-impact skid distance (D_1):

$$\begin{aligned} S_{1p} &= \sqrt{254\,D_1\,f} \\ &= \sqrt{30(32)(0.85)} \\ &= \sqrt{254(9.75)(0.85)} \\ &= 28.57 \text{ mph } (45.88 \text{ km/hr}) \end{aligned}$$

b. Pre-impact (lead-in) speed loss (S_{2p}) (English units) based on pre-impact skid distance (D_2):

$$S_{2p} = \sqrt{30\,D_2\,f}$$

Pre-impact (lead-in) speed loss (S_{2p}) (S.I. units) based on pre-impact skid distance (D_2):

$$\begin{aligned} S_{2p} &= \sqrt{254\,D_2\,f} \\ &= \sqrt{30(47)(0.85)} \\ &= \sqrt{254(14.33)(0.85)} \\ &= 34.62 \text{ mph } (55.70 \text{ km/hr}) \end{aligned}$$

c. The post-impact speed (S_3) of Vehicle 1 (English units) based on its post-impact skid distance is

$$S_3 = \sqrt{30\,D_3\,f}$$

The post-impact speed (S_3) of Vehicle 1 (SI units) based on its post-impact skid distance is

$$\begin{aligned} S_3 &= \sqrt{254\, D_3\, f} \\ &= \sqrt{30(16)(0.85)} \\ &= \sqrt{254(4.88)(0.85)} \\ &= 20.20 \text{ mph } (32.46 \text{ km/hr}) \end{aligned}$$

d. To determine the post-impact speed (S_4) of Vehicle 2, consider that both vehicles skidded the same post-impact distance. Thus, the post-impact speed of Vehicle 2 will be the same as that of Vehicle 1. Hence, S_1 = 32.46 km/hr (40 mph).

e. For the impact speed of Vehicle 1, we may assume from a reliable eyewitness that the impact speed of Vehicle 1 was 64 km/hr (40 mph). Hence, S_1 = 64 km/hr (40 mph).

f. The impact speed of Vehicle 2 is

$$\begin{aligned} S_2 &= \frac{W_1}{W_2} S_3 + S_4 + \frac{W_1}{W_2} S_1 \\ &= \frac{3500}{2000}(20.20) + 20.20 - \frac{3500}{2000}(40) \\ &= -14.45 \text{ mph} \end{aligned}$$

or

$$\begin{aligned} S_2 &= \frac{W_1}{W_2} S_3 + S_4 + \frac{W_1}{W_2} S_1 \\ &= \frac{1587.60}{907.20}(32.46) + 32.46 - \frac{1587.60}{907.20}(64) \\ &= -23.25 \text{ km/hr} \end{aligned}$$

g. The speed of Vehicle 1 at the beginning of its lead-in (pre-impact) skid marks is

$$\begin{aligned} S_{10} &= \sqrt{S_{1p}^2 + S_1^2} \\ &= \sqrt{28.57^2 + 40^2} \\ &= \sqrt{45.88^2 + 64^2} \\ &= 49.16 \text{ mph } (79.09 \text{ km/hr}) \end{aligned}$$

h. The speed of Vehicle 2 at the beginning of its lead-in (pre-impact) skid marks is

$$\begin{aligned} S_{20} &= \sqrt{S_{2p}^2 + S_2^2} \\ &= \sqrt{34.62^2 + (-14.45)^2} \\ &= \sqrt{55.70^2 + (-23.25)^2} \\ &= 37.51 \text{ mph } (60.36 \text{ km/hr}) \end{aligned}$$

Problem 10.23

A vehicle skids 28.04 m (92 ft) before running off a bridge. The drag factor is 0.63 with 85% braking efficiency. It is found that the center of mass of the vehicle falls 2.74 m (9 ft) vertically and has traveled 19.20 m (63 ft) horizontally. The takeoff angle of the vehicle was determined to be 12°. Determine the speed of the vehicle at the start of the skid and at the start of the vault.

Solution to Problem 10.23

When REC-TEC software is used, the speed of the vault is determined first as shown in Figure 10.23.

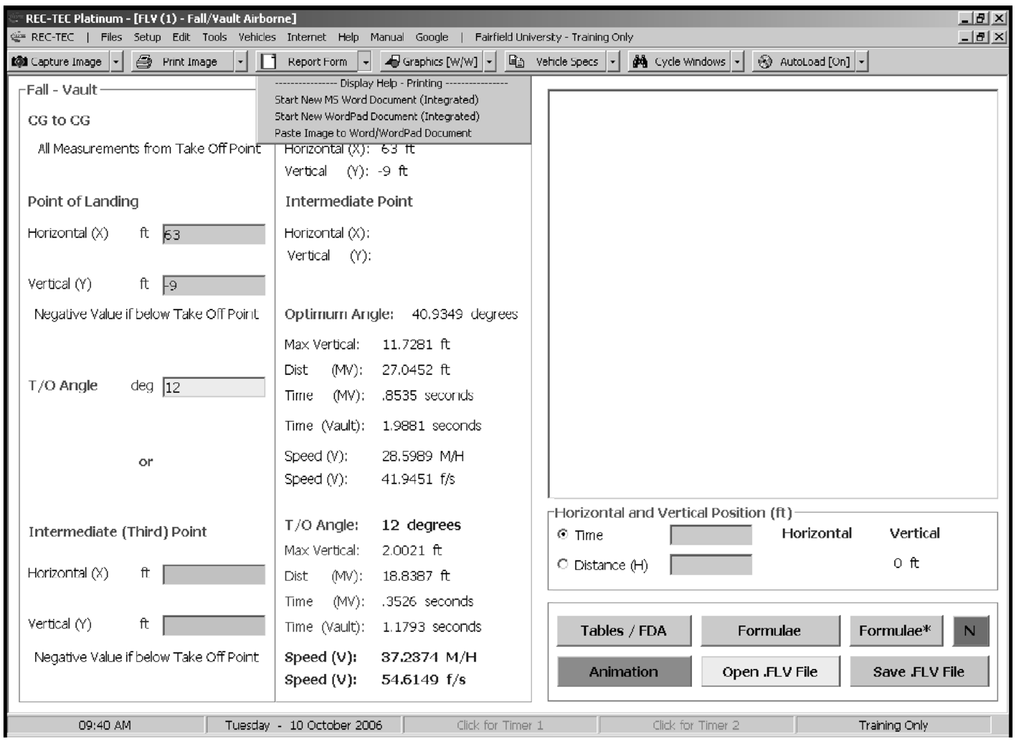

Figure 10.23 Results obtained with REC-TEC software.

The program displays information based on the skid, including the time and distance if this had been a skid to a stop.

Vault speed = 37.2374 mph

Speed at start of skid = 53.4934 mph

Problem 10.24

Determine the yaw speed with a chord of 16.76 m (55 ft) and a middle ordinate of 0.54 m (21 in.). The drag factor is 0.62, the super elevation is +0.1, and the track width of the vehicle is 1.52 m (60 in.).

Solution to Problem 10.24

With the use of REC-TEC software, the yaw speed is 48.0753 mph, as shown in Figure 10.24.

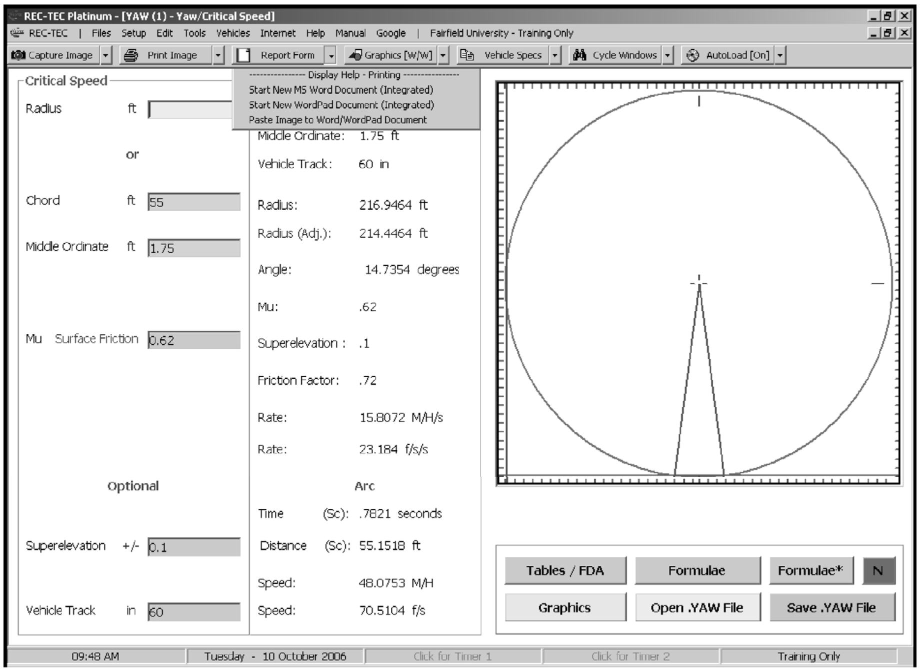

Figure 10.24 Results obtained with REC-TEC software.

Problem 10.25

Vehicle 1 travels from West to East and collides with Vehicle 2, which is traveling from South to North. The weights of Vehicles 1 and 2 are 1020.60 kg (2250 lb) and 1587.60 kg (3500 lb), respectively. After impact, Vehicle 1 travels 30° East of North for a distance of 18.288 m (60 ft) on a surface with a drag factor of 0.60. Vehicle 2 travels 60° East of North for a distance of 13.716 m (45 ft) on a surface with a drag factor of 0.70. Determine the speeds of the vehicles at impact.

Solution to Problem 10.25

With the use of REC-TEC software, the results obtained are shown in Figure 10.25.

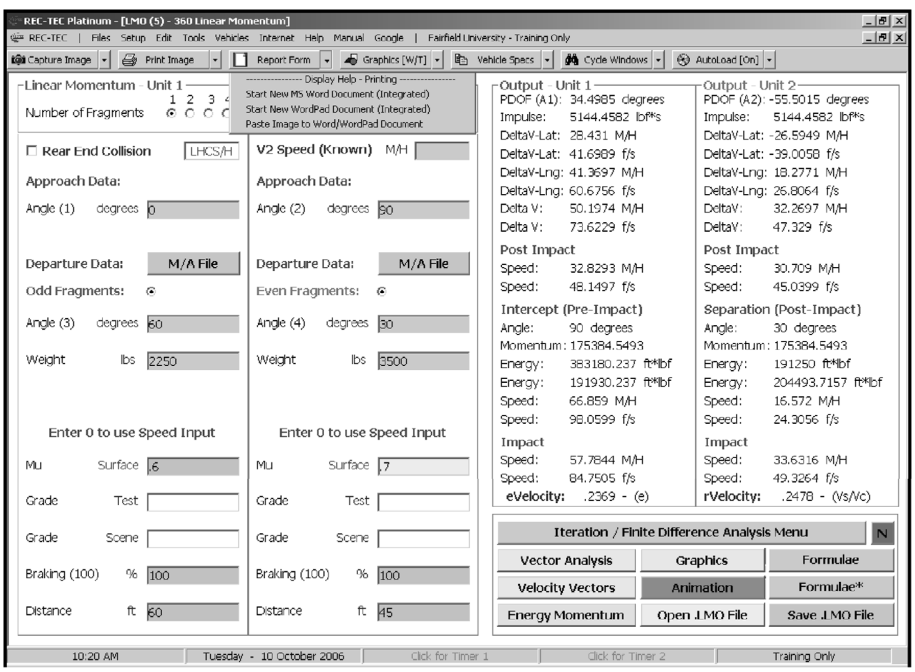

Figure 10.25 Results obtained with REC-TEC software.

Impact speed of Vehicle 1 = 57.7844 mph

Impact speed of Vehicle 2 = 33.6316 mph

Problem 10.26

Vehicle 2 was traveling at 96.54 km/hr (60 mph) when it was struck head-on by Vehicle 1. Neither vehicle moved appreciably after impact. Determine the ΔV for each of the two vehicles and the impact speed of Vehicle 1. The weight of Vehicle 1 is 1451.52 kg (3200 lb), and the weight of Vehicle 2 is 997.92 kg (2200 lb).

Solution to Problem 10.26

Because Vehicle 2 went from 96.54 km/hr (60 mph) at impact and was stopped instantly, its ΔV is equal to 96.54 km/hr (60 mph). We can determine ΔV for Vehicle 1 using the weight/ΔV relationship as

$$\frac{\Delta V_1}{\Delta V_2} = \frac{w_2}{w_1}$$

$$\frac{\Delta V_1}{96.54} = \frac{997.92}{1451.52}$$

Hence, ΔV_1 = 66.3 km/hr (41.25 mph).

Because the ΔV of Vehicle 1 was 66.37 km/hr (41.25 mph) and the vehicle was stopped instantly by the impact, its impact speed must be 66.37 km/hr (41.25 mph).

Problem 10.27

Vehicle 2 was slowing for a red traffic signal when it was struck squarely in the rear by Vehicle 1. After impact, both vehicles were traveling 64.36 km/hr (40 mph). Vehicle 2 received a ΔV of 32.18 km/hr (20 mph). Vehicle 2 received a ΔV of 27.35 km/hr (17 mph). Determine the impact speeds of the two vehicles.

Weight of Vehicle 1 = 2268 kg (5000 lb)

Weight of Vehicle 2 = 1587.60 kg (3500 lb)

Solution to Problem 10.27

Because Vehicle 2 was moving at 64.36 km/hr (40 mph) after impact and received a speed gain of 27.35 km/hr (17 mph) from the rear impact, its impact speed was

$$S_2 + 27.35 = 64.36$$

or

$$S_2 = 37.01 \text{ km/hr (16.79 mph)}$$

Here, we use the fact that the ΔV for each vehicle will be inversely proportional to the mass of the vehicles. Hence, the ΔV for Vehicle 1 was

$$\frac{\Delta V \text{ of Vehicle 1}}{\Delta V \text{ of Vehicle 2}} = \frac{\text{weight of Vehicle 2}}{\text{weight of Vehicle 1}}$$

$$\frac{\Delta V \text{ of Vehicle 1}}{27.35} = \frac{1587.60}{2268} = 0.70$$

or

$$\Delta V \text{ of Vehicle } 1 = 0.70(27.35) = 19.15 \text{ km/hr } (11.90 \text{ mph})$$

We now solve for the impact speed of Vehicle 1. Because the vehicle was moving at 64.36 km/hr (40 mph) after impact and lost 19.15 km/hr (11.90 mph), its impact speed was

$$S_2 - 19.15 = 64.36$$

or

$$S_2 = 83.51 \text{ km/hr } (51.90 \text{ mph})$$

Problem 10.28

Vehicle 1 is traveling westbound at 56.32 km/hr (35 mph), and Vehicle 2 is traveling southbound at 45.05 km/hr (28 mph). The two vehicles collide at an intersection. After impact, Vehicle 1 is traveling W 20° S at 43.44 km/hr (27 mph). After impact, Vehicle 2 is traveling W 55° S at 32.18 km/hr (20 mph). Determine the ΔV of each of the vehicles.

Weight of Vehicle 1 = 1587.60 kg (3500 lb)

Weight of Vehicle 2 = 1270.08 kg (2800 lb)

Solution to Problem 10.28

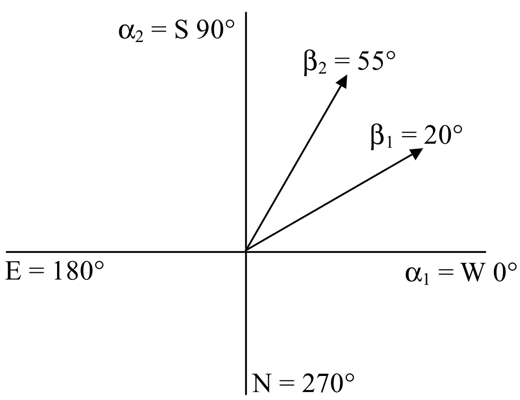

Figure 10.28 Vehicle directions.

Speed/direction of Vehicle 1 at impact: West @ 56.32 km/hr

Speed/direction of Vehicle 1 after impact: W 20º S @ 43.44 km/hr

Breaking post-impact movements into components gives

West component: 43.44 cos 20° = 40.82

South component: 43.44 sin 20° = 14.86

ΔV West: 56.32 – 40.82 = 15.5 km/hr (speed loss)

AutoLibrary

ΔV South: 0 + 14.86 = 14.86 km/hr (speed gain)

Vehicle 1 ΔV: $\sqrt{15.5^2 + 14.86^2} = 21.47$ km/hr (13.35 mph)

Speed/direction of Vehicle 2 at impact: South @ 45.05 km/hr

Speed/direction of Vehicle 2 after impact: W 55° S @ 32.18 km/hr

Breaking post-impact movements into components gives

West component: 32.18 cos 55° = 18.46

South component: 32.18 sin 55° = 26.36

ΔV West: 0 + 18.46 = 18.46 km/hr (11.47 mph) speed gain

ΔV South: 45.05 − 26.36 = 18.69 km/hr (11.62 mph) speed loss

Vehicle 2 ΔV: $\sqrt{18.46^2 + 18.69^2} = 26.27$ km/hr (16.33 mph)

$$\frac{\text{Vehicle 1 } \Delta V}{\text{Vehicle 2 } \Delta V} = \frac{21.47}{26.27} = 0.8173$$

$$\frac{w_2}{w_1} = \frac{1270.08}{1587.60} = 0.80$$

Problem 10.29

The following information is given for an accident involving two vehicles:

	Vehicle 1	Vehicle 1
Curb weight	2086.56 kg (4600 lb)	1451.52 kg (3200 lb)
Driver weight	99.79 kg (220 lb)	90.72 kg (200 lb)
Cargo weight	22.68 kg (50 lb)	18.14 kg (40 lb)
Travel direction at impact	West	South
Travel direction after impact	W 18° S	W 40° S
Post-impact travel distance	13.716 m (45 ft)	10.668 m (35 ft)
Post-impact drag factor	0.35	0.40

Determine the following:

a. The impact speed of Vehicle 1

b. The impact speed of Vehicle 2

c. The ΔV of Vehicle 1

d. The ΔV of Vehicle 2

Solution to Problem 10.29

We will begin by considering that West is our 0° line. All angles measured counter-clockwise will be considered positive.

	Direction	θ	sin θ	cos θ
α_1	West	0	0	1
α_2	South	90°	1	0
β_1	W 18º S	18°	0.309	0.951
β_2	W 40º S	40°	0.643	0.766

To determine the weight ratios, let $w_1 = 1$. Then,

$$w_2 = \frac{1560.38}{2209.03} = 0.7064$$

where

w_1 = weight of Vehicle 1 = 2086.56 + 99.79 + 22.68 = 2209.03 kg

w_2 = weight of Vehicle 2 = 1451.52 + 90.72 + 18.14 = 1560.38 kg

The post-impact speed of Vehicle 1 is

$$\begin{aligned} s_1 &= \sqrt{254\, d_1\, f_1} \\ &= \sqrt{254(13.716)(0.35)} \\ &= 34.92 \text{ km/hr (mph)} \end{aligned}$$

The post-impact speed of Vehicle 2 is

$$\begin{aligned} s_2 &= \sqrt{254\, d_2\, f_2} \\ &= \sqrt{254(10.668)(0.40)} \\ &= 32.92 \text{ km/hr (mph)} \end{aligned}$$

The law of conservation of momentum along the East-West axis is

$$S_1 w_1 \cos\alpha_1 + S_2 w_2 \cos\alpha_2 = s_1 w_1 \cos\beta_1 + s_2 w_2 \cos\beta_2$$

$$S_1 = (1)(1) + 0 = 34.92(1)(0.951) + 32.92(0.7064)(0.766)$$

$$S_1 = 33.209 + 17.813 = 51.02 \text{ km/hr (31.71 mph)}$$

The law of conservation of momentum along the North-South axis is

$$S_1 w_1 \sin\alpha_1 + S_2 w_2 \sin\alpha_2 = s_1 w_1 \sin\beta_1 + s_2 w_2 \sin\beta_2$$

$$0 + S_2(0.7064)(1) = 34.92(1)(0.309) + 32.92(0.7064)(0.643) = 10.79 + 14.95$$

or

$$S_2 = 36.44 \text{ km/hr (22.65 mph)}$$

The ΔV calculation gives

Speed/direction of Vehicle 1 at impact: West @ 51.02 km/hr

Speed/direction of Vehicle 1 after impact: W 18° S @ 34.92 km/hr

Breaking the post-impact moment into components gives

West component: 34.92 cos 18° = 33.21

South component: 34.92 sin 18° = 10.79

ΔV West: 51.02 – 33.21 = 17.81 km/hr (11.07 mph)

ΔV South: 0 + 10.79 = 10.79 km/hr (6.71 mph)

Vehicle 1 ΔV: $\sqrt{17.81^2 + 10.79^2} = 20.82$ km/hr (12.94 mph)

Speed/direction of Vehicle 2 at impact: South @ 36.44 km/hr

Speed/direction of Vehicle 2 after impact: W 40° S @ 32.92 km/hr

Breaking the post-impact moment into components gives

West component: 32.92 cos 40° = 25.22

South component: 32.92 sin 40° = 21.16

ΔV West: 0 + 25.22 = 25.22 km/hr (mph)

ΔV South: 36.44 – 21.16 = 15.28 km/hr (mph)

Vehicle 2 ΔV: $\sqrt{22.22^2 + 15.28^2} = 29.47$ km/hr (18.32 mph)

Check: The ΔV values should be in inverse proportion to the weights.

Hence,

$$\frac{\text{Vehicle 1 } \Delta V}{\text{Vehicle 2 } \Delta V} = \frac{w_2}{w_1}$$

$$\frac{w_2}{w_1} = \frac{1560.38}{2209.03} = 0.7064$$

$$\frac{\text{Vehicle 1 } \Delta V}{\text{Vehicle 2 } \Delta V} = \frac{20.82}{29.47} = 0.7064$$

With REC-TEC software, the results obtained are shown in Figure 10.29.

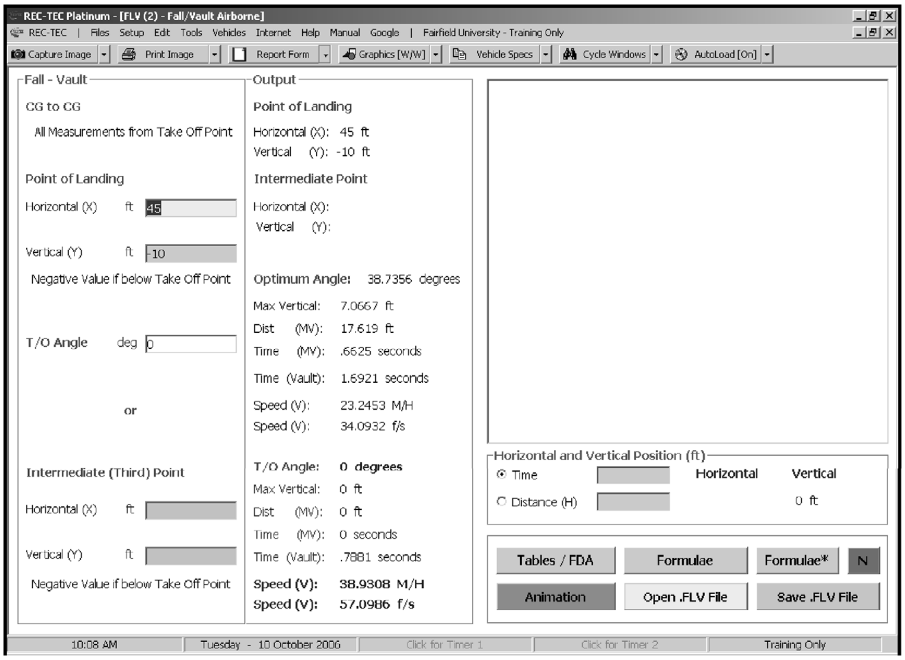

Figure 10.29 Results obtained with REC-TEC software.

Problem 10.30

The following information is known about an accident involving two vehicles (see Figure 10.30):

Weight of the Ford vehicle: 1587.60 kg (3500 lb)

Weight of the Chevrolet vehicle: 1360.80 kg (3000 lb)

The Ford skids 21.336 m (70 ft) after impact at a drag factor of f = 0.44.

The Chevrolet skids 23.4696 m (77 ft) after impact at a drag factor of f = 0.44.

The maximum acceleration of the Chevrolet is 16 ft/s^2 (4.88 m/s^2).

The distance from the stop line to the point of impact is 9.754 m (32 ft).

Determine the following:

a. The impact speed of the Ford

b. The impact speed of the Chevrolet

c. Could the Chevrolet have stopped at the Stop sign?

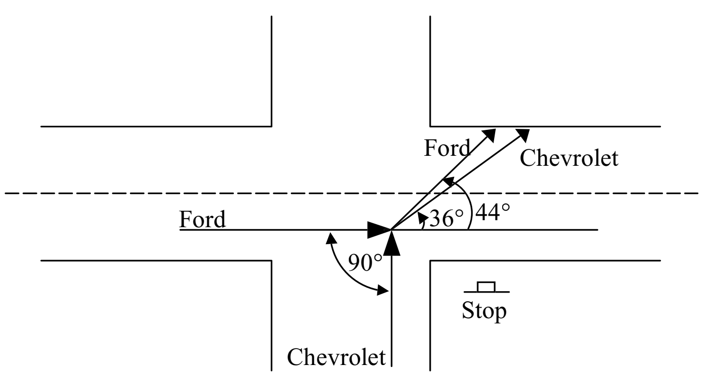

Figure 10.30 Driving directions.

Solution to Problem 10.30

We will begin setting by assuming that the impact direction of Vehicle 1 is our 0° line and will call that direction East. All angles measured counterclockwise will be considered positive.

	Direction	θ	sin θ	cos θ
α_1	East	0	0	1
α_2	North	90°	1	0
β_1	E 44° N	44°	0.695	0.719
β_2	E 36° N	36°	0.588	0.809

To determine the weight ratios, let $w_1 = 1$. Then,

$$w_2 = \frac{1360.80}{1587.60} = 0.857$$

The post-impact speed of Vehicle 1 is

$$\begin{aligned} s_1 &= \sqrt{254 d_1 f_1} \\ &= \sqrt{254(21.336)(0.44)} \\ &= 48.83 \text{ km/hr } (30.35 \text{ mph}) \end{aligned}$$

The post-impact speed of Vehicle 2 is

$$\begin{aligned} s_2 &= \sqrt{254 d_2 f_2} \\ &= \sqrt{254(23.4696)(0.45)} \\ &= 51.79 \text{ km/hr } (32.19 \text{ mph}) \end{aligned}$$

a. Applying the law of conservation along the East-West axis gives

$$S_1 w_1 \cos\alpha_1 + S_2 w_2 \cos\alpha_2 = s_1 w_1 \cos\beta_1 + s_2 w_2 \cos\beta_2$$

$$S_1 = (1)(1) + 0 = 48.83(1)(0.719) + 51.79(0.857)(0.809)$$

$$S_1 = 35.109 + 35.907 = 71.02 \text{ km/hr } (44.14 \text{ mph})$$

b. Applying the law of conservation of momentum along the North-South axis gives

$$S_1 w_1 \sin\alpha_1 + S_2 w_2 \sin\alpha_2 = s_1 w_1 \sin\beta_1 + s_2 w_2 \sin\beta_2$$

$$\begin{aligned} 0 + S_2(0.857)(1) &= 48.83(1)(0.695) + 51.79(0.857)(0.588) \\ &= 33.937 + 26.098 = 60.035 \end{aligned}$$

or

$$S_2 = 70.05 \text{ km/hr } (43.54 \text{ mph})$$

c. Let us check the maximum acceleration of the Chevrolet (a = 4.88 m/s^2) against the distance from the stop line to impact (d = 9.754 m), and see whether the vehicle could have reached impact speed at that distance.

$$d = vt + \frac{1}{2}at^2$$

$$9.754 = 0t + \frac{1}{2}(4.88)t^2$$

or

$$t = 2 \text{ s}$$

The fastest the Chevrolet could have been going if it stopped would have been

$$V = at = 4.88(2) = 9.76 \text{ m/s} = 35.10 \text{ km/h}$$

Hence, the answer is No.

With REC-TEC software, the results obtained are shown in Figure 10.30(a).

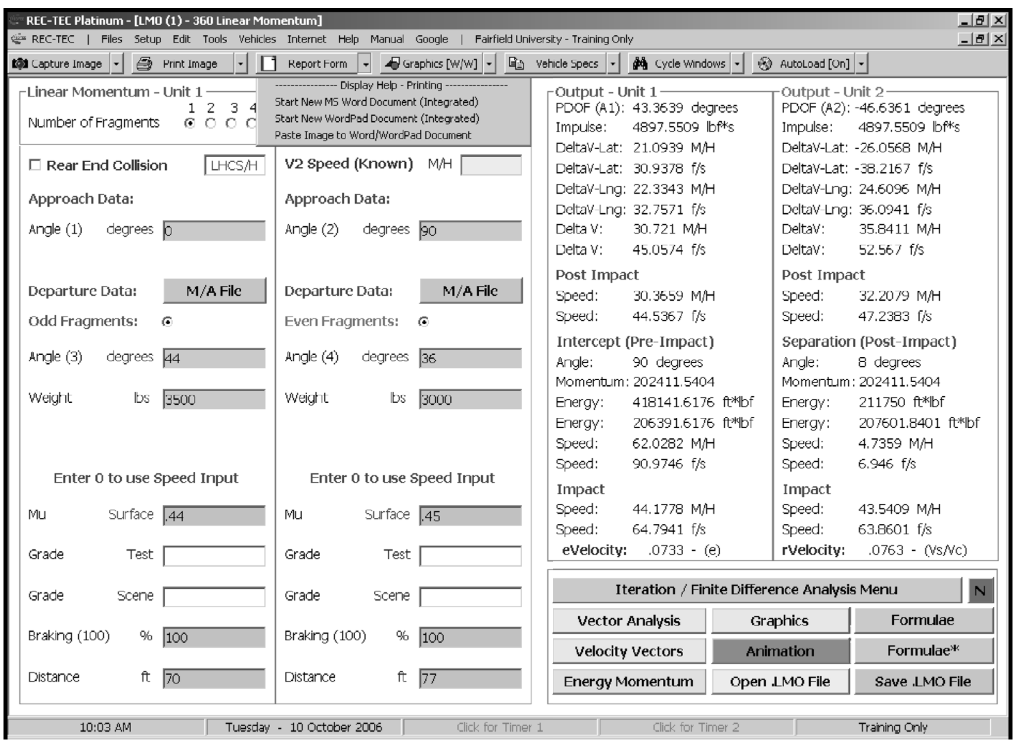

Figure 10.30 (a) Results obtained with REC-TEC software.

Problem 10.31

Vehicle 2 weighed 4250 lb (1928 kg) and was stopped at a traffic signal when it was struck in the rear by a 3330-lb (1510-kg) Eastbound Vehicle 1. After impact, Vehicle 1 traveled approximately East 53 ft (16 m) at an average drag factor of 0.38. After impact, Vehicle 2 traveled East 58 ft (17.5 m) at an average drag factor of 0.38. Calculate the impact speed of Vehicle 1.

Solution to Problem 10.31

The solution to this problem can be reached in two steps. First, we will calculate the equivalent barrier speed for the damage on each vehicle. Then, we will use the dissipation of energy equation to solve for the impact speed of Vehicle 1. In addition to the previous data, we will use the following variables:

C_{AVG1} = Average crush depth to the front of Vehicle 1 = 15 in.

C_{AVG2} = Average crush depth to the rear of Vehicle 2 = 25 in.

ebs_1 = Equivalent barrier speed of Vehicle 1

ebs_2 = Equivalent barrier speed of Vehicle 2

The equivalent barrier speed of Vehicle 1 is

$$ebs_1 = 1.4 * C_{AVG1} + 7 = 1.4 * 13 + 7 = 25.2$$

The equivalent barrier speed of Vehicle 2 is

$$ebs_2 = 1.15 * C_{AVG2} + 5 = 1.15 * 25 + 5 = 33.7$$

Substituting and solving the dissipation of energy equation gives

$$\begin{aligned} S_1^2 * W_1 + S_2^2 * W_2 &= s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 \\ &\quad + ebs_2^2 * W_2 S_1^2 * 1 + 0^2 * 1.27 \\ &= 24.6^2 * 1 + 25.7^2 * 1.27 + 25.2^2 * 1 \\ &\quad + 33.7^2 * 1.27 \end{aligned}$$

$$S_1^2 = 605 + 839 + 635 + 1442 = 3521$$

$$S_1 = \sqrt{3521} = 59.34 \text{ or } 59 \text{ mph } (95 \text{ km/hr})$$

Problem 10.32

For the collision in Problem 10.31, recalculate the impact speed of Vehicle 1 using dissipation of energy. Vehicle 1 has an average of 13 in. (33 cm) of crush on its front. Vehicle 2 has an average of 25 in. (63 cm) of crush on its rear. Use the following Campbell equations:

S in mph, C_{AVG} in in.:	Vehicle 1 front:	$ebs = 1.40{*}C_{MAX} + 7$
	Vehicle 2 rear:	$ebs = 1.15{*}C_{MAX} + 5$
S in kph, C_{AVG} in cm:	Vehicle 1 front:	$ebs = 0.89{*}C_{MAX} + 11$
	Vehicle 2 rear:	$ebs = 0.73{*}C_{MAX} + 8$

Solution to Problem 10.32

This is a simple application of the in-line (one-dimensional) conservation of momentum formula. The variables are as follows:

S_1 = Speed of Vehicle 1 at impact

S_2 = Speed of Vehicle 2 at impact = 0 mph

s_1 = Speed of Vehicle 1 after impact

s_2 = Speed of Vehicle 2 after impact

d_1 = Post-impact skid distance of Vehicle 1 = 53 ft

d_2 = Post-impact skid distance of Vehicle 1 = 58 ft

f_1 = Drag factor of Vehicle 1 = 0.38

f_2 = Drag factor of Vehicle 2 = 0.38

W_1 = Weight of Vehicle 1 = 3330 lb

W_2 = Weight of Vehicle 2 = 4250 lb

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 4250/3330 = 1.27$$

To determine the speeds after impact, we use the post-impact skid distance and the basic skid formula. For Vehicle 1,

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 53 * 0.38} = 24.6 \text{ mph}$$

The speed after impact of Vehicle 2 is

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 58 * 0.38} = 25.7 \text{ mph}$$

The conservation of momentum formula gives

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 + 0 * 1.27 = 24.6 * 1 + 25.7 * 1.27$$

$$S_1 = 24.6 + 32.7$$

$$S_1 = 57.3 \text{ or } 57 \text{ mph } (92 \text{ km/hr})$$

Problem 10.33

A 2006 Ford F-250 pickup (Vehicle 1) collides head-on with a 2002 Ford Focus (Vehicle 2). Neither vehicle moves significantly after impact. The weights are 6408 lb (2907 kg) for the F-250 and 3425 lb (1554 kg) for the Focus. Use the averages of the following front crush data (various heights), as well as the following Campbell equations:

Vehicle 1 Average Crush Depth:

Hood:	0 in. (0 cm)
Top Bumper:	4.3 in. (10.9 cm)
Bottom Bumper:	7.2 in. (18.3 cm)

Vehicle 2 Average Crush Depth:

Hood:	41.9 in. (106.4 cm)	
Top Bumper:	20.9 in. (53.1 cm)	
Bottom Bumper:	17.3 in. (43.9 cm)	

S in mph, C_{AVG} in in.:	Vehicle l front:	$ebs = 1.54*C_{MAX} + 7$
	Vehicle 2 front:	$ebs = 1.68*C_{MAX} + 7$
S in km/hr, C_{AVG} in cm:	Vehicle l front:	$ebs = 0.97*C_{MAX} + 11$
	Vehicle 2 front:	$ebs = 1.06*C_{MAX} + 11$

Calculate the impact speeds of both vehicles.

Solution to Problem 10.33

We will ultimately solve this problem using conservation of momentum and dissipation of energy. The parameters are as follows:

S_1 = Speed of Vehicle 1 at impact

s_1 = Speed of Vehicle 1 after impact = 0 mph

S_2 = Speed of Vehicle 2 at impact

s_2 = Speed of Vehicle 2 after impact = 0 mph

W_1 = Weight of Vehicle l = 6408 lb

W_2 = Weight of Vehicle 2 = 3425 lb

C_{AVG1} = Average crush depth to Vehicle l = $(0 + 4.3 + 7.2)/3 = 3.8$ in.

C_{AVG2} = Average crush depth to Vehicle 2 = $(41.9 + 20.9 + 17.3)/3 = 26.7$ in.

ebs_1 = Equivalent barrier speed of Vehicle l

ebs_2 = Equivalent barrier speed of Vehicle 2

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 3425/6408 = 0.534$$

The equivalent barrier speed of Vehicle 1 crush is

$$ebs_1 = 1.54 * C_{1AVG} + 7 = 1.54 * 3.8 + 7 = 12.9 \text{ mph}$$

The equivalent barrier speed of Vehicle 2 crush is

$$ebs_2 = 1.68 * C_{2AVG} + 7 = 1.68 * 26.7 + 7 = 51.9 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that the term for the impact momentum of Vehicle 2 is negative because that vehicle is heading

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in the opposite direction of Vehicle 1.

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 - S_2 * 0.534 = 0 * 1 + 0 * 0.534$$

$$S_1 - S_2 * 0.534 = 0$$

$$S_1 = S_2 * 0.534$$

Substituting and solving the dissipation of energy equation gives

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

$$S_1^2 * 1 + S_2^2 * 0.534 = 0^2 * 1 + 0^2 * 0.534 + 12.9^2 * 1 + 51.9^2 * 0.534$$

$$S_1^2 + S_2^2 * 0.534 = 166 + 1438 = 1604$$

Substituting S_2*0.534 for S_1 gives

$$\left(0.534 * S_2\right)^2 + S_2^2 * 0.534 = 1604$$

$$0.285 * S_2^2 + S_2^2 * 0.534 = S_2^2 * 0.819 = 1604$$

$$S_2 = \sqrt{1958} = 44.25 \text{ or } 44 \text{ mph } (71 \text{ km/hr})$$

Substituting this into the momentum equation gives

$$S_1 - S_2 * 0.534 = 44.25 * 0.534 = 23.63 \text{ or } 24 \text{ mph } (39 \text{ km/hr})$$

This scenario was National Highway Traffic Safety Administration (NHTSA) Crash Test #5683. The actual impact speeds were 24.0 mph for Vehicle 1 and 43.9 mph for Vehicle 2. The preceding Campbell equations were derived from the actual barrier tests for these years, makes, and models of vehicles. The approach of averaging crush depths, measured at the top and bottom of the bumper plus the hood, yielded much more accurate results for the partial underride impact than the two approaches evaluated.

Problem 10.34

A 2006 Ford F-250 pickup (Vehicle1) collides head-on with a 2002 Ford Focus (Vehicle 2). Neither vehicle moves significantly after impact. The F-250 has an average of 4.3 in. (11 cm) of crush on the top of its front bumper. Vehicle 2 has an average of 20.9 in. (53 cm) of crush on the top of its front bumper. The weights are 6408 lb (2907 kg) for the F-250 and 3425 lb (1554 kg) for the Focus. Use the following Campbell equations:

S in mph, C_{AVG} in in.:	Vehicle 1 front:	ebs = 1.53*C_{MAX} + 7
	Vehicle 2 front:	ebs = 1.68*C_{MAX} + 7
S in km/h, C_{AVG} in cm:	Vehicle 1 front:	ebs = 0.97*C_{MAX} + 11
	Vehicle 2 front:	ebs = 1.06*C_{MAX} + 11

Calculate the impact speeds of both vehicles.

Solution to Problem 10.34

We ultimately will solve this problem using conservation of momentum and dissipation of energy. The parameters are as follows:

S_1 = Speed of Vehicle 1 at impact

s_1 = Speed of Vehicle 1 after impact = 0 mph

S_2 = Speed of Vehicle 2 at impact

s_2 = Speed of Vehicle 2 after impact = 0 mph

W_1 = Weight of Vehicle 1 = 6408 lb

W_2 = Weight of Vehicle 2 = 3425 lb

C_{AVG1} = Average crush depth to Vehicle 1 = 4.3 in.

C_{AVG2} = Average crush depth to Vehicle 2 = 20.9 in.

ebs_1 = Equivalent barrier speed of Vehicle 1

ebs_2 = Equivalent barrier speed of Vehicle 2

The equivalent barrier speed of Vehicle 1 crush is

$$ebs_1 = 1.54 * C_{1AVG} + 7 = 1.54 * 4.3 + 7 = 13.6 \text{ mph}$$

The equivalent barrier speed of Vehicle 2 crush is

$$ebs_2 = 1.68 * C_{2AVG} + 7 = 1.68 * 20.9 + 7 = 42.1 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that the term for the impact momentum of Vehicle 2 is negative because it is heading in the opposite direction of Vehicle 1.

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 - S_2 * 0.534 = 0 * 1 + 0 * 0.534$$

$$S_1 - S_2 * 0.534 = 0$$

$$S_1 = S_2 * 0.534$$

Substituting and solving the dissipation of energy equation gives

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

$$S_1^2 * 1 + S_2^2 * 0.534 = 0^2 * 1 + 0^2 * 0.534 + 13.6^2 * 1 + 42.1^2 * 0.534$$

$$S_1^2 + S_2^2 * 0.534 = 185 + 946 = 1131$$

Substituting S_2*0.534 for S_1 gives

$$\left(0.534 * S_2\right)^2 + S_2^2 * 0.534 = 1131$$

$$0.285 * S_2^2 + S_2^2 * 0.534 = S_2^2 * 0.819 = 1131$$

$$S_2 = \sqrt{1381} = 37.15 \text{ or } 37 \text{ mph } (60 \text{ km/hr})$$

Substituting this into the momentum equation gives

$$S_1 - S_2 * 0.534 = 37.15 * 0.534 = 19.84 \text{ or } 20 \text{ mph } (32 \text{ km/hr})$$

This scenario was NHTSA Crash Test #5683. The actual impact speeds were 24.0 mph for Vehicle l and 43.9 mph for Vehicle 2. Because the above Campbell equations were derived from the actual barrier tests for these years, makes, and models of vehicles, the speed underestimation likely was due to the partial underride impact that resulted in more crush on the grill and hood of Vehicle 1 and the lower bumper of Vehicle 2. In the next problem, we will take this crush into account.

Problem 10.35

The F-250 in Problem 10.34 had an average of 7.2 in. (18 cm) of crush at the lower bumper, whereas the Focus suffered 41.9 in. (106 cm) of crush at the hood. Recalculate the speed of the vehicles by averaging these crush values with the top-of-bumper crush numbers for these vehicles.

Solution to Problem 10.35

We ultimately will solve this problem by using the same approach. However, this time, we will attempt to account for the underride by the crush depth on the top and bottom bumpers of Vehicle 1, and on the top of the bumper averaged with crush depth at the top of the hood for Vehicle 2. The new parameters are as follows:

C_{AVG1} = Average crush depth to Vehicle l = $\left(4.3 + 7.2\right)/2 = 5.75$ in.

C_{AVG2} = Average crush depth to Vehicle 2 = $\left(20.9 + 41.9\right)/2 = 31.4$ in.

The equivalent barrier speed of Vehicle l crush is

$$ebs_1 = 1.54 * C_{1AVG} + 7 = 1.54 * 5.75 + 7 = 15.9 \text{ mph}$$

The equivalent barrier speed of Vehicle 2 crush is

$$ebs_2 = 1.68 * C_{2AVG} + 7 = 1.68 * 31.4 + 7 = 59.7 \text{ mph}$$

The momentum results will be the same as those in Problem 10.37. Substituting and solving the dissipation of energy equation gives

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

$$S_1^2 * 1 + S_2^2 * 0.534 = 0^2 * 1 + 0^2 * 0.534 + 15.9^2 * 1 + 59.7^2 * 0.534$$

$$S_1^2 + S_2^2 * 0.534 = 253 + 1903 = 2156$$

Substituting S_2*0.534 for S_1 gives

$$\left(0.534 * S_2\right)^2 + S_2^2 * 0.534 = 2156$$

$$0.285 * S_2^2 + S_2^2 * 0.534 = S_2^2 * 0.819 = 2156$$

$$S_2 = \sqrt{2632} = 51.3 \text{ or } 51 \text{ mph (83 km/hr)}$$

Substituting this into the momentum equation gives

$$S_1 - S_2 * 0.534 = 51.3 * 0.534 = 27.4 \text{ or } 27 \text{ mph (44 km/hr)}$$

These results are higher than the actual impact speeds of 24.0 mph for Vehicle 1 and 43.9 mph for Vehicle 2. This is not surprising because this model assumes that half of the frontal strength of Vehicle 2 is in the hood. Another possible approach would be to average together damage at the top of the bumper, the bottom of the bumper, and the hood.

Problem 10.36

While crossing a public street, a golf cart containing only the driver is struck broadside by a 2004 Chevrolet Cavalier. The vehicles remain together after impact, traveling 87 ft (26.5 m) before coming to rest. The total weights of the vehicles, occupants, and cargo are 1107 lb (502 kg) for the golf cart and 2953 lb (1340 kg) for the Cavalier. Use 0.65 for the post-impact drag factor of the golf cart and 0.30 for the post-impact drag factor of the Cavalier. Calculate the impact speed of the Cavalier.

Solution to Problem 10.36

Because the much smaller golf cart is moving at a low speed, we may apply the in-line (one-dimensional) conservation of momentum formula. We will refer to the Cavalier as Vehicle 1 and the golf cart as Vehicle 2. The variables are as follows:

S_1 = Speed of Vehicle 1 at impact

S_2 = Speed of Vehicle 2 at impact = 0 mph

s_1 = Speed of Vehicle 1 after impact

s_2 = Speed of Vehicle 2 after impact

d_1 = Vehicle 1 post-imact skid distance = 18 ft

d_2 = Vehicle 2 post-impact skid distance = 22 ft

f_1 = Drag factor of Vehicle 1 = 0.30

f_2 = Drag factor of Vehicle 2 = 0.65

W_1 = Weight of Vehicle l = 3800 lb

W_2 = Weight of Vehicle 2 = 2700 lb

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 1107/2953 = 0.375$$

We must be careful to use the weighted average drag factor. If we were to calculate independent slide-to-stop values for both vehicles, we would, in effect, be mixing momentum and energy, and that would throw off our results somewhat. The weighted average drag factor is

$$f = \left(f_1 * W_1 + f_2 * W_2\right) / \left(W_1 + W_2\right)$$

$$f = \left(0.30 * 1 + 0.65 * 0.375\right) / \left(1 + 0.375\right)$$

$$f = 0.395$$

Now we can use the post-impact skid distance and the basic skid formula. For both vehicles, this is

$$s_1 = \sqrt{30 * d * f} = \sqrt{30 * 87 * 0.38} = 32.1 \text{ mph}$$

The conservation of momentum formula is as follows:

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 + 0 * 0.375 = 32.1 * 1 + 32.1 * 0.375$$

$$S_1 = 44.1 \text{ or } 44 \text{ mph (71 km/hr)}$$

Problem 10.37

For the accident in Problem 10.36, the post-impact drag factors for the vehicles are unknown. However, the front of the Cavalier was measured and was found to have an average crush of 4.75 in. (12 cm). An NHTSA compliance crash of a 2004 Cavalier into a full-length rigid barrier at 29.6 mph (47.6 km/hr) showed that the car suffered 15.3 in. (38.9 cm) of frontal crush. Recalculate the speed of the Cavalier, using a 7-mph (11-km/hr) damage threshold for the car and assuming that the energy absorbed by the front of the car equals the energy absorbed by the damage to the golf cart.

Solution to Problem 10.37

In an engineering sense, we have four unknown factors: the impact speed of the Cavalier, the post-impact speed of both vehicles, and the energy absorbed by the damage to the golf cart. As in Problem 10.36, we can assume that the impact speed of the golf cart is zero.

We need four independent equations to solve for the four unknowns. The momentum and energy equations are two of those equations. The third equation is that the post-impact speeds of the two vehicles are equal. The fourth equation can come from setting the absorbed energies of the two vehicles equal to each other, as instructed in the problem. We expect this approach to yield a conservative result because the fiberglass body of the golf cart probably absorbed much more energy than the front of the Cavalier.

For the 2004 Cavalier, the Campbell model, in equation form, is

$$ebs = b_1 * C_{AVG} + b_0$$

where

$$ebs = \text{Equivalent barrier speed (mph)} = 29.6 \text{ mph}$$

$$b_1 = \text{Stiffness constant (mph/in.)}$$

$$C_{AVG2} = \text{Average crush depth} = 15.3 \text{ in.}$$

$$b_0 = \text{Damage threshold constant, given as 7 mph in this problem}$$

The equation also can be generated for metric units. The b_1 and b_0 would have to be different than in the imperial unit equation.

Assume that $b_0 = 7$ mph. Then,

$$ebs = b_1 * C_{AVG} + b_0$$

Therefore,

$$29.6 = b_1 * 15.3 + 7$$
$$22.6 = b_1 * 15.3$$
$$1.48 = b_1$$

Therefore,

$$ebs = 1.48 * C_{AVG} + 7$$

The ebs for the front of the Cavalier is

$$ebs = 1.48 * 4.75 + 7 = 14.0 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that the term for the impact momentum of Vehicle 2 is considered zero because the golf cart is much lighter, is traveling at a low speed, and is heading at a right angle to Vehicle 1. The post-impact speeds are the same but unknown. We will define as "s" the following:

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 - 0 * 0.375 = s * 1 + s * 0.375$$

$$S_1 = 1.375 * s$$

The dissipation of energy equation is

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

The post-crash speeds are the same, and the energies absorbed by the crush on the vehicles are the same. The equation simplifies to

$$S_1^2 * W_1 + 0^2 * W_2 = s^2 * W_1 + s^2 * W_2 + ebs_1^2 * W_1 + ebs_1^2 * W_1$$

$$S_1^2 * 1 + 0^2 * 0.375 = s^2 * 1 + s^2 * 0.375 + 14.0^2 * 1 + 14.0^2 * 1$$

$$S_1^2 = 1.375 * s^2 + 392$$

Substituting (s*1.375) S1 gives

$$(1.375 * s)^2 = 1.375 * s^2 + 392$$

$$1.89 * s^2 = 392$$

$$0.515 * s^2 = 392$$

$$s = \sqrt{760} - 27.6$$

Substituting this into the momentum equation gives

$$S_1 = s * 1.375 = 27.6 * 1.375 = 37.9 \text{ or } 38 \text{ mph (61 km/h)}$$

Problem 10.38

Vehicle 1 weighs 3650 lb (1656 kg) and collides head-on with Vehicle 2, which weighs 3100 lb (1406 kg). Both vehicles bounce back a negligible distance and stop. The Campbell equations and crash measurements for the vehicles are as follows:

	Vehicle 1	Vehicle 2
Campbell equation (mph, in.)	ebs = 1.32*C_{AVG} + 7	ebs = 1.41*C_{AVG} +7
Campbell equation (km/hr, cm)	ebs = 0.84*C_{AVG} + 11	ebs = 0.89*C_{AVG} + 11
C_1	20 in. (51 cm)	17 in. (43 cm)
C_2	21.5 in. (55 cm)	18 in. (46 cm)
C_3	22 in. (56 cm)	18.5 in. (47 cm)
C_4	22.5 in. (57 cm)	18 in. (46 cm)
C_5	22 in. (56 cm)	16.5 in. (42 cm)
C_6	18 in. (46 cm)	15 in. (38 cm)

Determine the impact speeds of both vehicles.

Solution to Problem 10.38

We ultimately will solve this problem by using conservation of momentum and dissipation of energy. The parameters are as follows:

S_1 = Speed of Vehicle 1 at impact

s_1 = Speed of Vehicle 1 after impact = 0 mph

S_2 = Speed of Vehicle 2 at impact

s_2 = Speed of Vehicle 2 after impact = 0 mph

W_1 = Weight of Vehicle l = 3650 lb

W_2 = Weight of Vehicle 2 = 3100 lb

C_{AVG1} = Average crush depth to Vehicle l front

C_{AVG2} = Average crush depth to Vehicle 2 rear

ebs_1 = Equivalent barrier speed of Vehicle 1

ebs_2 = Equivalent barrier speed of Vehicle 2

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 3100/3650 = 0.849$$

We must estimate the energies dissipated by crush via determination of their equivalent barrier impact speeds. The average crush to the front of Vehicle 1 is

$$C_{1avg} = (C_1/2 + C_2 + C_3 + C_4 + C_5 + C_6/2)/5$$

$$C_{1avg} = (20/2 + 21.5 + 22 + 22.5 + 22 + 18/2)/5 = 21.4 \text{ in.}$$

The equivalent barrier speed is

$$ebs_1 = 1.32 * C_1 + 7 = 1.32 * 21.4 + 7 = 35.2 \text{ mph}$$

Average crush to the front of Vehicle 2 is

$$C_{2avg} = (17/2 + 18 + 18.5 + 18 + 16.5 + 15/2)/5 = 17.4 \text{ in.}$$

The crush energy equivalent speed is

$$ebs_2 = 1.41 * C_{2avg} + 7 = 1.41 * 17.4 + 7 = 31.5 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that the term for the impact momentum of Vehicle 2 is negative because it is heading in the opposite direction of Vehicle 1.

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 - S_2 * 0.849 = 0 * 1 + 0 * 0.849$$

$$S_1 - S_2 * 0.849 = 0$$

$$S_1 = S_2 * 0.849$$

Substituting and solving the dissipation of energy equation gives

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

$$S_1^2 * 1 + S_2^2 * 0.849 = 0^2 * 1 + 0^2 * 0.849 + 35.2^2 * 1 + 31.5^2 * 0.849$$

$$S_1^2 * 1 + S_2^2 * 0.849 = 1239 + 842 = 2081$$

Substituting $S_2 * 0.849$ for S_1 gives

$$(0.849 * S_2)^2 + S_2^2 * 0.849 = 2081$$

$$0.781 * S_2^2 + S_2^2 * 0.849 = S_2^2 * 1.570 = 2081$$

$$S_2 = \sqrt{1326} = 36.4 \text{ or } 36 \text{ mph (59 km/hr)}$$

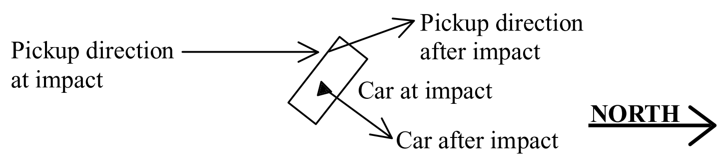
Substituting this into the momentum equation gives

$$S_1 = S_2 * 0.849 = 36.4 * 0.849 = 30.9 \text{ or } 31 \text{ mph (50 km/hr)}$$

Problem 10.39

A car (Vehicle 1) makes a left turn into the path of a pickup truck (Vehicle 2). The pickup truck driver sees the car turning in front of him but cannot avoid striking the rear quarter panel of the car. The impact spins the car in a clockwise fashion. Use the following data to calculate the impact speed of both vehicles and the speed of the pickup truck at the start of the pre-crash skid:

	Vehicle 1	Vehicle 2
Curb weight:	3019 lb (1369 kg)	4550 lb (2064 kg)
Driver weight:	117 lb (53 kg)	217 lb (98 kg)
Cargo weight:	65 lb (29 kg)	120 lb (54 kg)
Pre-crash skid distance:	—	53 ft (16.2 m)
Pre-crash skid drag factor:	—	0.75
Direction at impact:	S 70° E	North
Direction after impact:	E 45° N	N 2° W
Distance after impact:	30 ft (9.1 m)	41 ft (12.5 m)
Drag factor after impact:	0.55	0.75



Solution to Problem 10.39

We have a conservation of momentum problem here. The variables are as follows:

S = Speed of Vehicle 2 at the start of pre-crash braking

S_1 = Speed of Vehicle 1 at impact

s_1 = Speed of Vehicle 1 after impact

S_2 = Speed of Vehicle 2 at impact

s_2 = Speed of Vehicle 2 after impact

W_1 = Weight of Vehicle 1 = 3019 + 117 + 65 = 3201 lb

W_2 = Weight of Vehicle 2 = 4550 + 217 + 120 = 4887 lb

d_1 = Post-impact travel distance of Vehicle 1 = 30 ft

d_2 = Post-impact travel distance of Vehicle 2 = 41 ft

d_{2PRE} = Pre-impact skid distance of Vehicle 2 = 53 ft

f_1 = Post-impact drag factor of Vehicle 1 = 0.55

f_2 = Post-impact drag factor of Vehicle 2 = 0.75

f_{2PRE} = Pre-impact braking drag factor of Vehicle 2 = 0.75

a_1 = Travel angle of Vehicle l at impact

a_2 = Travel angle of Vehicle 2 at impact

b_1 = Travel angle of Vehicle 1 after impact

b_2 = Travel angle of Vehicle 2 after impact

We will start setting by assuming that North is our 0° line. All angles measured counterclockwise will be considered as positive. We will tabulate our angles and then determine the trigonometric values based on the following:

	Direction	Q	sin q	cos q
a_1:	S 70° E	250°	–0.940	–0.342
a_2:	North	0°	0	1
b_1:	E 45° N	315°	–0.707	0.707
b_2:	N 2° W	2°	0.035	0.999

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 4887/3201 = 1.527$$

The post-impact speed of Vehicle 1 is

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 30 * 0.55} = 22.2 \text{ mph}$$

The post-impact speed of Vehicle 2 is

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 41 * 75} = 30.4 \text{ mph}$$

The conservation of momentum formula along the East-West axis is

$$S_1 * W_1 \sin a_1 + S_2 * W_2 * \sin a_2 = s_1 * W_1 * \sin b_1 + s_2 * W_2 * \sin b_2$$

$$S_1 * 1 - 0.940 + 1.527 * 0 = 22.2 * 1 - 0.707 + 30.4 * 1.527 * 0.035$$

$$S_1 = 14.7 \text{ or } 15 \text{ mph } (24 \text{ km/hr})$$

The conservation of momentum along the North-South axis is

$$S_1 * W_1 \cos a_1 + S_2 * W_2 * \cos a_2 = s_1 * W_1 * \cos b_1 + s_2 * W_2 * \cos b_2$$

$$14.7 * 1 - 0.342 + S_2 * 1.527 * 1 = 22.1 * 1 * 0.707 + 30.4 * 1.527 * 0.999$$

$$-5.03 + 1.527 * S_2 = 15.7 + 46.4 = 62.1$$

$$1.527 * S_2 = 62.1 + 5.03 = 67.1$$

$$S_2 = 43.9 \text{ or } 44 \text{ mph } (71 \text{ km/hr})$$

The speed of Vehicle 2 at the start of pre-crash braking is

$$S = \sqrt{S_2^2 + 30 * d_{2PRE} * f_{2PRE}}$$

$$S = \sqrt{43.9^2 + 30 * 53 * 0.75}$$

$$S = \sqrt{927 + 1192} = \sqrt{3119}$$

$$S = 55.85 \text{ or } 56 \text{ mph } (90 \text{ km/hr})$$

Problem 10.40

A 2000 Nissan Maxima skids 47 ft (14.3 m) on dry asphalt (drag factor = 0.82) and strikes the side of a stationary train blocking a grade crossing. The impact is to one of the solid iron "trucks" (i.e., wheels, suspension) of a gondola car in the train. The Maxima rebounds a short distance and comes to rest. The front crush profile of the Maxima was measured and compared with an NHTSA front barrier crash test as follows:

	NHTSA Crash Test	Grade Crossing Accident
Impact speed:	35.2 mph (56.4 km/hr)	???
C_1	18.3 in. (46.5 cm)	20.7 in. (52.6 cm)
C_2	20.0 in. (50.8 cm)	27.9 in. (70.9 cm)
C_3	20.0 in. (50.8 cm)	25.5 in. (64.8 cm)
C_4	20.5 in. (52.1 cm)	25.5 in. (64.8 cm)
C_5	20.5 in. (52.1 cm)	23.3 in. (59.2 cm)
C_6	18.4 in. (46.7 cm)	20.5 in. (52.1 cm)

Treat the train as a fixed rigid barrier. Derive the Campbell equation for the Maxima and use it to calculate the impact speed of the car. Also, determine the speed of the Maxima at the start of its pre-crash skid.

Solution to Problem 10.40

We ultimately will solve this problem using the Campbell equation and the combined speed formula. We will use the following variables:

C_{AVG} = Average crush depth to Nissan front (in.)

ebs = Equivalent barrier speed of Nissan (mph)

d = Pre-crash skid distance of Nissan = 47 ft

f = Drag factor of Nissan during pre-crash skid = 0.82

We must estimate the energies dissipated by crush via determination of the equivalent barrier impact speed of the Nissan. First, we must calibrate the Campbell equation from the NHTSA New Car Assessment Program (NCAP) crash test. The average crush to the front of the 2000 Nissan Maxima tested by NHTSA at 35.2 mph was as follows:

$$C_{AVG} = \left(C_1/2 + C_2 + C_3 + C_4 + C_5 + C_6/2\right)/5$$

$$C_{AVG} = \left(18.3/2 + 20.0 + 20.0 + 20.5 + 20.5 + 18.4/2\right)/5 = 19.9 \text{ in.}$$

Assume the following:

$$b_0 = 7 \text{ mph}$$

$$ebs_1 = b_1 * C_{AVG} + b_0$$

$$35.2 = b_1 * 19.9 + 7$$

$$28.2 = b_1 * 19.9$$

$$1.42 = b_1$$

The Campbell equation for a 2000 Nissan Maxima is

$$ebs = 1.42 * C_{AVG} + 7$$

The average crush to the front of the Nissan from the train impact is

$$C_{AVG} = \left(20.7/2 + 27.3 + 25.5 + 25.5 + 23.3 + 20.5/2\right)/5 = 24.6 \text{ in.}$$

Because we are treating the stationary train as a fixed rigid barrier and the Nissan struck it hit head-on with negligible post-impact movement, the equivalent barrier speed will equal the impact speed (S_1) of the Nissan as follows:

$$S_1 = ebs = 1.42 * C_{AVG} + 7 = 1.4 * 24.6 + 7 = 41.9 \text{ or } 42 \text{ mph (67 km/hr)}$$

The speed of the car at the start of the pre-crash skid (S) is determined by using the combined speed formula

$$S = \sqrt{S_1^2 + 30 * d * f}$$

$$S = \sqrt{41.9^2 + 30 * 47 * 0.82}$$

$$S = \sqrt{1756 + 1156} = \sqrt{2912}$$

$$S = 53.96 \text{ or } 54 \text{ mph (87 km/hr)}$$

Problem 10.41

Vehicle 1 weighs 3400 lb (1542 kg) and collides head-on with Vehicle 2, which weighs 2750 lb (1247 kg). Both vehicles bounce back a negligible distance and stop. The Campbell equations and crush measurements for the vehicles are as follows:

	Vehicle 1	Vehicle 2
Campbell equation (mph, in.)	$ebs = 1.32*C_{AVG} + 7$	$ebs = 1.41*C_{AVG} + 7$
Campbell equation (km/hr, cm)	$ebs = 0.84*C_{AVG} + 11$	$ebs = 0.89*C_{AVG} + 11$
C_1	30 in. (76 cm)	27 in. (69 cm)
C_2	31.5 in. (80 cm)	28 in. (71 cm)
C_3	32 in. (81 cm)	28.5 in. (72 cm)
C_4	32.5 in. (83 cm)	28 in. (71 cm)
C_5	32 in. (81 cm)	26.5 in. (67 cm)
C_6	28 in. (71 cm)	25.5 in. (64.8 cm)

Determine the impact speeds of both vehicles.

Solution to Problem 10.41

We ultimately will solve this problem by using conservation of momentum and dissipation of energy. The parameters are as follows:

S_1 = Speed of Vehicle 1 at impact

s_1 = Speed of Vehicle 1 after impact = 0 mph

S_2 = Speed of Vehicle 2 at impact

s_2 = Speed of Vehicle 2 after impact = 0 mph

W_1 = Weight of Vehicle l = 3400 lb

W_2 = Weight of Vehicle 2 = 2750 lb

C_{AVG1} = Average crush depth to Vehicle l front

C_{AVG2} = Average crush depth to Vehicle 2 rear

ebs_1 = Equivalent barrier speed of Vehicle l

ebs_2 = Equivalent barrier speed of Vehicle 2

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 2750/3400 = 0.809$$

We must estimate the energies dissipated by crush via determination of their equivalent barrier impact speeds. The average crush to the front of Vehicle 1 is

$$C_{1avg} = \left(C_1/2 + C_2 + C_3 + C_4 + C_5 + C_6/2\right)/5$$

$$C_{1avg} = \left(30/2 + 31.5 + 32 + 32.5 + 32 + 28/2\right)/5 = 31.4 \text{ in.}$$

The equivalent barrier speed is

$$ebs_1 = 1.32 * C_{1AVG} + 7 = 1.32 * 31.4 + 7 = 48.4 \text{ mph}$$

The average crush to the front of Vehicle 2 is

$$C_{2avg} = \left(27/2 + 28 + 28.5 + 28 + 26.5 + 25/2\right)/5 = 27.4 \text{ in.}$$

The crush energy equivalent speed is

$$ebs_2 = 1.41 * C_{2avg} + 7 = 1.41 * 27.4 + 7 = 45.6 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that the term for the impact momentum of Vehicle 2 is negative because Vehicle 2 is heading in the opposite direction of Vehicle l.

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 - S_2 * 0.809 = 0 * 1 + 0 * 0.809$$

$$S_1 - S_2 * 0.809 = 0$$

$$S_1 = S_2 * 0.809$$

Substituting and solving the dissipation of energy equation gives

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

$$S_1^2 * 1 + S_2^2 * 0.809 = 0^2 * 1 + 0^2 * 0.809 + 48.4^2 * 1 + 45.6^2 * 0.809$$

$$S_1^2 * 1 + S_2^2 * 0.809 = 2343 + 1682 = 4025$$

Substituting $S_2 * 0.809$ for S_1 gives

$$(0.809 * S_2)^2 + S_2^2 * 0.809 = 4025$$

$$0.654 * S_2^2 + S_2^2 * 0.809 = S_2^2 * 1.464 = 4025$$

$$S_2 = \sqrt{2749} = 52.44 \text{ or } 52 \text{ mph (84 km/hr)}$$

Substituting this into the momentum equation gives

$$S_1 = S_2 * 0.809 = 52.44 * 0.809 = 42.4 \text{ or } 42 \text{ mph (68 km/hr)}$$

Problem 10.42

An Eastbound 2002 Dodge Ram 1500 collides head-on with a Westbound 2004 Honda Accord and pushes the Accord straight backward. Given the following data, determine the impact speeds of both vehicles:

	Dodge	Honda
Curb weight:	4952 lb (2246 kg)	3217 lb (1459 kg)
Driver weight:	264 lb (120 kg)	211 lb (96 kg)
Cargo weight:	300 lb (136 kg)	150 lb (68 kg)
Damage width:	Entire front	Entire front
Crush point C_1:	3.1 in. (8 cm)	17.4 in. (44 cm)
Crush point C_2:	3.8 in. (9.5 cm)	25.7 in. (65 cm)
Crush point C_3:	8.8 in. (22.5 cm)	27.6 in. (70 cm)
Crush point C_4:	8.5 in. (21.5 cm)	26.7 in. (68 cm)
Crush point C_5:	4.0 in. (10 cm)	24.8 in. (63 cm)
Crush point C_6:	7.5 in. (19 cm)	22.4 in. (58 cm)
Post-crash travel:	81 ft (24.6 m) East	81 ft (24.6 m) East
Drag factor:	0.50	0.50

Solution to Problem 10.42

We were given that the Dodge pickup was subjected to a 35.10-mph impact and suffered 18.0 in. of crush. We also learned that the Honda was subjected to a 34.8-mph impact and suffered 20.7 in. of crush. We now derive the Campbell equations for both vehicles:

For the Dodge,

$$ebs = b_1 * C_{AVG1} + b_0$$

Assume that $b_0 = 7$ mph. Then,

$$35.0 = b_1 * 18.0 + 7$$
$$28.0 = b_1 * 18.0$$
$$1.56 = b_1$$

Therefore, the equation for the Dodge is

$$ebs = 1.56 * C_{AVG1} + 7$$

For the Honda,

$$ebs = b_1 * C_{AVG2} + b_0$$

Assume that $b_0 = 7$ mph. Then,

$$34.8 = b_1 * 20.7 + 7$$
$$27.8 = b_1 * 20.7$$
$$1.34 = b_1$$

Therefore, the equation for the Honda is

$$ebs = 1.34 * C_{AVG2} + 7$$

We ultimately will solve this problem by using conservation of momentum and dissipation of energy. The parameters are as follows:

S_1 = Speed of the Dodge at impact

s_1 = Speed of the Dodge after impact

S_2 = Speed of the Honda at impact

s_2 = Speed of the Honda after impact

W_1 = Weight of the Dodge = 4952 + 264 + 300 = 5516 lb

W_2 = Weight of the Honda = 3217 + 211 + 150 = 3578 lb

C_{AVG1} = Average crush depth to the front of the Dodge

C_{AVG2} = Average crush depth to the front of the Honda

ebs_1 = Equivalent barrier speed of the Dodge

ebs_2 = Equivalent barrier speed of the Honda

d = Post-impact travel distance for both vehicles = 81 ft

t = Post-impact drag factor for both vehicles = 0.50

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 3578/5516 = 0.649$$

We must estimate the energies dissipated by crush via determination of their equivalent barrier impact speeds. The average crush to the front of the Dodge is

$$C_{1avg} = \left(C_1/2 + C_2 + C_3 + C_4 + C_5 + C_6/2\right)/5$$

$$C_{1avg} = \left(3.1/2 + 3.8 + 8.8 + 8.5 + 4.0 + 7.5/2\right)/5 = 6.1 \text{ in.}$$

The equivalent barrier speed is

$$ebs_1 = 1.56 * C_{AVG1} + 7 = 1.56 * 6.1 + 7 = 16.5 \text{ mph}$$

The average crush to the front of the Honda is

$$C_{2avg} = \left(17.9/2 + 25.7 + 27.6 + 26.7 + 24.8 + 22.4/2\right)/5 = 249 \text{ in.}$$

The crush energy equivalent speed is

$$ebs_2 = 1.34 * C_{AVG2} + 7 = 1.34 * 24.9 + 7 = 40.4 \text{ mph}$$

The speeds after impact are

$$s = \sqrt{30 * d * f} = \sqrt{30 * 81 * 0.5} = 40.4 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that the term for impact momentum of the Honda is negative because it is heading in the opposite direction of the Dodge.

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 - S_2 * 0.649 = 34.6 * 1 + 34.6 * 0.649$$

$$S_1 - S_2 * 0.649 = 57.0$$

$$S_1 = S_2 * 1.069 + 57.0$$

Substituting and solving the dissipation of energy equation gives

$$S_1^2 * W_1 + S_2^2 * W_2 = s_1^2 * W_1 + s_2^2 * W_2 + ebs_1^2 * W_1 + ebs_2^2 * W_2$$

$$S_1^2 * 1 + S_2^2 * 0.649 = 34.6^2 * 1 + 34.6^2 * 0.649 + 16.5^2 * 1 + 40.4^2 * 0.649$$

$$S_1^2 + S_2^2 * 0.649 = 1497 + 778 + 272 + 1059 = 3306$$

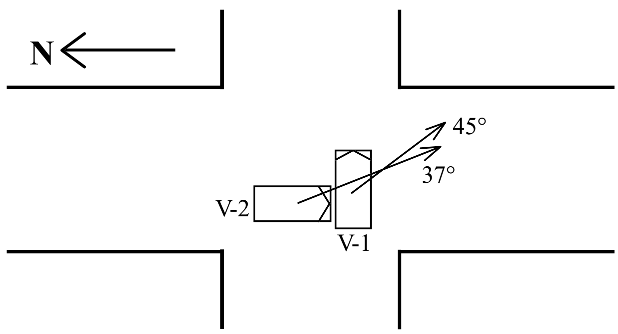
If both the momentum and energy equations are solved by trial and error, starting with $S_2 = 0$ and $S_1 = 57.0$ mph for the momentum equation and 57.5 mph for the energy equation, this tells us that the Dodge was traveling approximately 57 mph at impact, while the Honda was stopped or on the verge of stopping at impact.

This problem was based on an NHTSA crash test in which a stopped Honda Accord was struck head-on by a Dodge Ram 1500 traveling at 57.4 mph.

Problem 10.43

Vehicle 1 is traveling Eastbound when it is struck broadside at an intersection by Southbound Vehicle 2. Both vehicles spin out of control and strike a curb on the southeast side of the intersection. Vehicle 1 continues over the curb and strikes a pole rear-first and comes to rest. Little damage to the pole occurs, but the rear of Vehicle 1 sustains a maximum of 21 in. (56 cm) of crush. Given the following data, calculate the speeds of Vehicles 1 and 2 at impact. Use the equation from Problem 4 to determine the speed of Vehicle 1 at impact with the pole. Use 10 mph (16 km/hr) equivalent energy speed for each vehicle to account for the curb impacts.

	Vehicle 1	**Vehicle 2**
Curb weight:	2874 lb (1304 kg)	3719 lb (1687 kg)
Driver weight:	210 lb (95 kg)	160 lb (73 kg)
Passengers' weight:	—	70 lb (32 kg)
Cargo weight:	55 lb (25 kg)	100 lb (45 kg)
Travel direction at impact:	East	South
Travel direction after impact:	S 45° E	S 37° E
Travel distance post-impact:	68 ft (20.7 m)	106 ft (32.3 m)
Drag factor:	0.53	0.38



Solution to Problem 10.43

We have a conservation of momentum problem here. The variables are as follows:

S_1 = Speed of Vehicle 1 at impact with Vehicle 2

s_1 = Speed of Vehicle 1 after impact

S_p = Speed of Vehicle 1 at impact with pole

S_c = Equivalent energy speed loss of vehicle impacts into curb = 10 mph

S_2 = Speed of Vehicle 2 at impact with Vehicle 1

s_2 = Speed of Vehicle 2 after impact

W_1 = Weight of Vehicle 1 = 2874 + 210 + 55 = 3139 lb

W_2 = Weight of Vehicle 2 = 3719 + 160 + 70 + 100 = 4049 lb

d_1 = Post-impact travel distance of Vehicle 1 = 68 ft

d_2 = Post-impact travel distance of Vehicle 2 = 106 ft

f_1 = Post-impact drag factor of Vehicle 1 = 0.53

f_2 = Post-impact drag factor of Vehicle 2 on guard rail = 0.38

C_{MAX} = Maximum crush to the rear of Vehicle 1 = 21 in.

α_1 = Travel angle of Vehicle 1 at impact

α_2 = Travel angle of Vehicle 2 at impact

β_1 = Travel angle of Vehicle 1 after impact

β_2 = Travel angle of Vehicle 2 after impact

We will start setting by assuming that South is our 0° line. All angles measured counterclockwise will be considered positive. We will tabulate our angles and then determine the trigonometric values as follows:

	Direction	**θ**	**sin θ**	**cos θ**
α_1:	East	90°	1	0
α_2:	South	0°	0	1
β_1:	S 45° E	45°	0.707	0.707
β_2:	S 37° E	37°	0.602	0.799

The weight ratios are

$$\text{If } W_1 = 1,\ W_2 = 4049/3139 = 1.29$$

Because there is no significant post-impact movement to consider, finding the impact speed of Vehicle 1 with the pole is easy. Simply substitute the 21 in. for C_{MAX} in the equation, and crunch the numbers as follows:

$$S_p = 0.55 * C_{MAX} + 3.9$$

$$S_p = 0.55 * 21 + 3.9 = 11.55 + 3.9 = 15.4 \text{ mph}$$

The speed at the start of post-impact skid can be solved by using the combined speed formula of

$$s_1 = \sqrt{S_p^2 + S_C^2 + 30 * d_1 * f_1}$$
$$= \sqrt{15.4^2 + 10^2 + 30 * 68 * 0.53}$$
$$= 37.7 \text{ mph}$$

We can use a similar approach for the post-impact speed of Vehicle 2 as

$$s_2 = \sqrt{S_C^2 + 30 * d_2 * f_2}$$
$$= \sqrt{10^2 + 30 * 106 * 0.38}$$
$$= 36.2 \text{ mph}$$

The conservation of momentum formula along the East-West axis is

$$S_1 * W_1 \sin\alpha_1 + S_2 * W_2 * \sin\alpha_2 = s_1 * W_1 * \sin\beta_1 + s_2 * W_2 * \sin\beta_2$$

$$S_1 1 * 1 + S_2 * 1.29 * 0 = 37.7 * 1 * 0.707 + 36.2 * 1.29 * 0.602$$

$$S_1 = 26.65 + 28.11$$

$$S_1 = 54.76 \text{ or } 55 \text{ mph (88 km/hr)}$$

The conservation of momentum along the North-South axis is

$$S_1 * W_1 \cos\alpha_1 + S_2 * W_2 * \cos\alpha_2 = s_1 * W_1 * \cos\beta_1 + s_2 * W_2 * \cos\beta_2$$

$$54.8 * 1 * 0 + S_2 * 1.29 * 1 = 37.7 * 1 * 0.707 + 36.2 * 1.29 * 0.799$$

$$0 + 1.29 * S_2 = 26.65 + 37.31 = 63.96$$

$$S_2 = 49.58 \text{ or } 50 \text{ mph } (80 \text{ km/hr})$$

Problem 10.44—VCRware Case Studies

VCRware—Vehicle Crash Reconstruction Software

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VCRware is a suite of 11 computer programs based in the Microsoft® Excel® (Microsoft Corporation, Redmond, Washington) environment. The programs provide the accident reconstructionist with the capability to reconstruct and analyze most vehicular and pedestrian accidents. This software suite is unique in the industry, in that the theoretical basis for each of the programs is covered in full technical detail in the book *Vehicle Accident Analysis and Reconstruction Methods*, by Raymond M. Brach, PhD, PE, and R. Matthew Brach, PhD, PE, published by SAE International in 2005.

The programs benefit greatly from the availability of analysis utilities resident in Microsoft Excel. These utilities include Goal Seek and Solver. Goal Seek allows the user to determine the value of an input parameter to achieve a specified or desired value of some spreadsheet output. For instance, the user can use Goal Seek with the Stopping Distance of a Vehicle spreadsheet to determine, in one step, the frictional drag coefficient needed to stop a vehicle within a given distance, starting from any initial speed. Solver is similar in many ways to Goal Seek but can handle more complex problems such as maximization and minimization with multiple variables. An example of this capability is the use of Solver in a crash analysis to find the pre-impact speeds and the coefficient of restitution that match the reconstructed post-impact speeds.

Two principal capabilities of any software used for the reconstruction of a vehicle-to-vehicle collision are the capability to model the impact between two vehicles (while the vehicles are in contact) and the capability to model the motion of the individual vehicles before and after impact (while the vehicles are not in contact). Included in VCRware are an impact model based on planar impact mechanics and a vehicle dynamics model capable of simulating the motion of a vehicle, with or without a semi-trailer, under all varieties of motion that a reconstructionist may encounter. The mechanics of both models have been validated with experimental data.

The impact model provided in VCRware is available in two varieties: (1) a standard version that models the impact between two vehicles, and (2) a version that models the impact between two vehicles, either or both of which may be articulated. The latter model, although complex, permits the modeling of collisions that involve one or two tractor semi-trailers for which the articulation of one or both of the vehicles is significant.

Two case studies are presented here to illustrate the capabilities of the planar impact and vehicle dynamics simulation programs. The first case study shows the use of the planar impact program that utilizes the ΔV data from an event data recorder (EDR). The second case study demonstrates the use of vehicle dynamics simulation in the post-impact motion of a damaged vehicle.

VCRware Case Study 1: Planar Impact Mechanics

Collision Description: A pickup truck and a sedan collide with the geometry as shown in Figure 10.31(a). The output from the EDR in the pickup truck, Vehicle 1, is shown in Figure 10.44(b). As determined from the EDR data, the longitudinal ΔV of the pickup truck as a result of the impact was 15 mph. The following information applies [see Figure 10.44(c)]:

$W_1 = 4000$ lb

$W_2 = 2800$ lb

$I_1 = 2500$ ft-lb/s^2

$I_2 = 1400$ ft-lb/s^2

$d_1 = 4.5$ ft

$d_2 = 2.1$ ft

$\theta_1 = -90°$

$\theta_2 = -135°$

$\Gamma = -45°$

$\varphi_1 = 20°$

$\varphi_2 = 60°$

$e = 0.2$

$\mu = \mu_0$

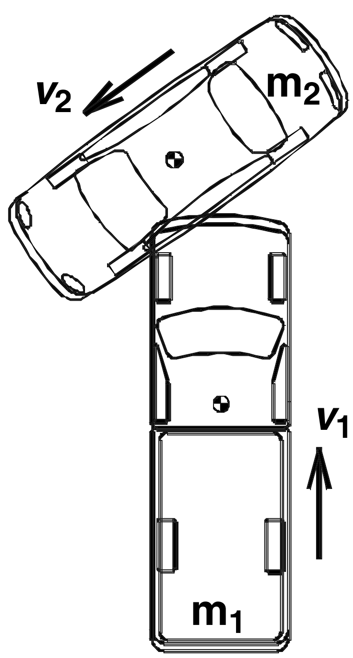


Figure 10.44 (a) Collision configuration for Case Study 1.

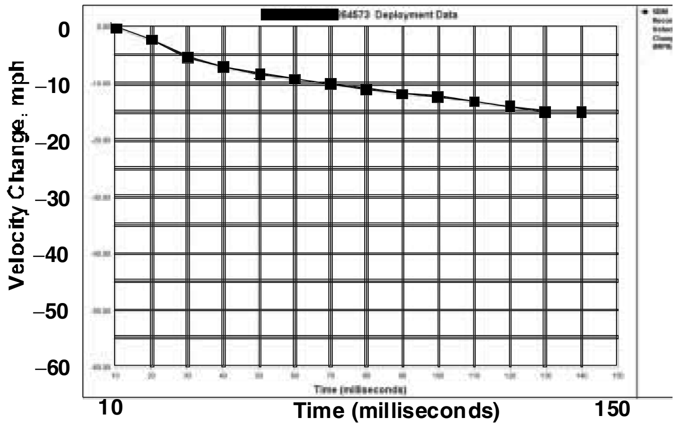


Figure 10.44 (b) Speed change data from the EDR download.

It is known that the sedan, Vehicle 2, was pulling out of a parking lot across the lane in which the pickup truck was traveling. The speed of the sedan at impact is estimated to be between 10 and 20 mph ($10 \leq v_2 \leq 20$ mph). This speed range was determined with the use of constant acceleration equations, given that the vehicle stopped before entering the roadway and using a suitable acceleration range over the distance from stop to impact. The rest positions of the vehicles are unknown because the police made no measurements and took no photographs during their investigation. Therefore, use of the speed of the vehicles via analysis of the post-impact motion or crush energy is not feasible. However, planar impact mechanics can be used to determine the pre-impact speed of the pickup truck while satisfying the ΔV criteria retrieved from the EDR. It is assumed that the vehicles reached a common velocity in the tangential direction ($\mu = \mu_0$) along the contact plane.

Speed Reconstruction: With data provided in the statement of the problem with the parameter assignments as shown in Figure 10.44(c), the data can be placed into the spreadsheet implementation of the planar impact mechanics impact model as shown in Figure 10.44(d). Note that the cell to the right of the cell labeled "ΔV_{EDR} =" contains the 15-mph ΔV that was retrieved from the EDR. The formula behind that cell calculates the longitudinal ΔV of Vet 1 by using the following equation:

$$\Delta V_{1y} = \Delta V_1 \cos\left(PDOF_1\right)$$

The use of the Goal Seek utility determines the speed of the pickup truck for the given set of input values and conditions such that the longitudinal ΔV of Vet 1, the pickup truck, is 15 mph. This calculation can be determined for a speed of the sedan of 20 mph [as shown in Figure 10.44(d)] and, separately, for an initial speed of Vet 2 of 10 mph. The corresponding range of pre-impact speeds of the pickup truck is 21 to 26 mph. If the coefficient of restitution is set at $e = 0$ rather than $e = 0.2$, the pre-impact speed range of the pickup truck changes to 23 to 29 mph. Additional analyses can be done to assess the uncertainty associated with other parameters such as the relative orientation of the vehicles, the location of the impact center C [see Figure 10.44(c)], or other parameters that may be of interest to the reconstructionist.

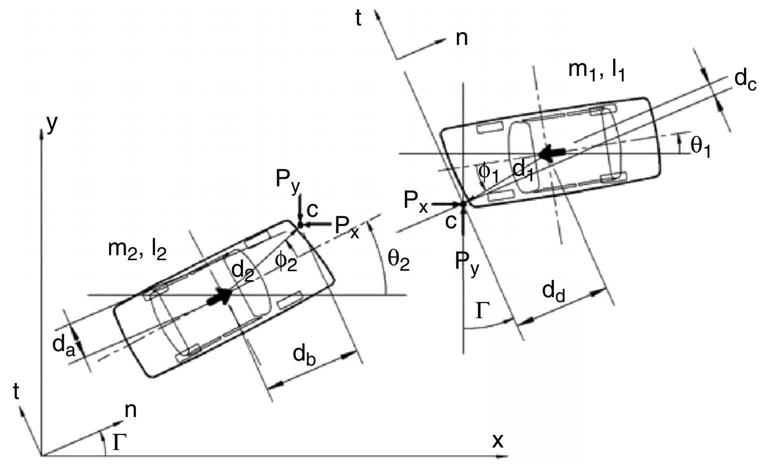


Figure 10.44 (c) Free body diagram of the planar impact of two vehicles.

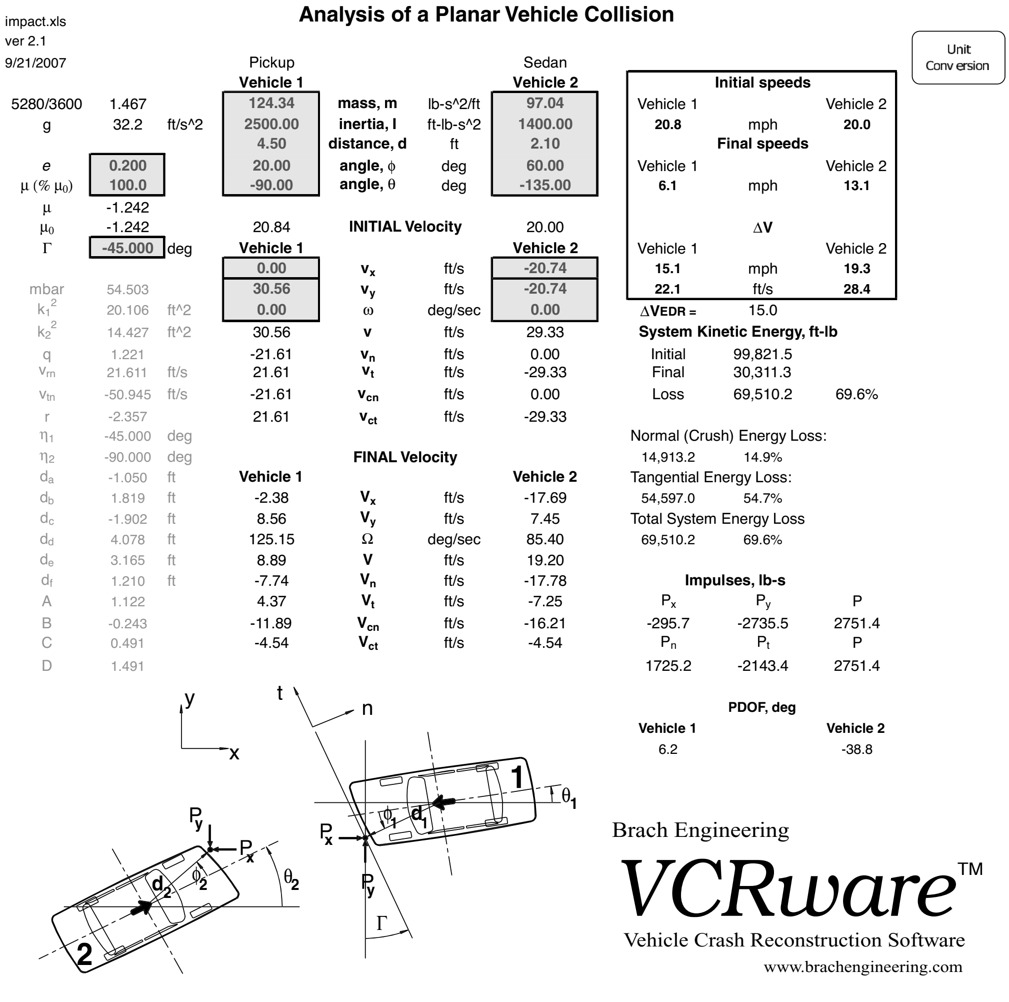


Figure 10.44 (d) Results of the planar impact analysis.

VCRware Case Study 2: Vehicle Dynamics

Vehicle dynamic simulation in the field of accident reconstruction frequently requires that special characteristics of a vehicle should be specified. For example, in the simulation of the post-impact motion of a vehicle, collision damage may prevent one of the wheels from rotating during its post-impact travel, or the wheelbase on one side of the vehicle may be changed as the result of damage. Under such circumstances, the simulation program must be able to accommodate changes to the individual wheels, to accurately represent the conditions under which the vehicle was moving. The following example illustrates this type of analysis by using the vehicle dynamics program vdynXL from VCRware.

Scenario Description: A stationary vehicle was hit in the rear in an offset front-to-rear collision. The impact damage to the vehicle has shifted the right rear wheel forward and against the wheel well, thereby shortening the right wheelbase and shifting the wheel laterally. The contact between the wheel and the wheel well locks the wheel from rotating during the post-impact motion. During the simulation, the vehicle dynamics program must allow locking of the right rear wheel and must accommodate an asymmetric wheelbase. A separate analysis of the impact between the vehicles based on planar impact mechanics gives the post-impact velocity of the vehicle at 45 mph and an initial counterclockwise angular velocity of 100°/s. The entire post-impact motion of the vehicle takes place on a level and dry asphalt surface. The following information applies:

W = 4057 lb

I = 2972.7 ft-lb/s^2

Left wheelbase: 9.6 ft

Right wheelbase: 9.2 ft

Front track: 5.26 ft

Rear track: 5.50 ft

Simulation Results: The data provided in the statement of the problem with vehicle parameters illustrated in Figure 10.44(e) can be placed into the vdynXL spreadsheet, as shown in Figure 10.44(f). Note in Figure 10.44(g) that the quantities L_3 and L_4, the longitudinal distances of the left rear and right rear wheels relative to the center of mass, respectively, are different. This reflects the asymmetry of the post-impact wheelbases. Figure 10.44(g) also shows that the parameter s_4, the longitudinal wheel slip of the right rear wheel, is set to $s_4 = 1$, indicating that the right rear wheel is locked. The motion of the vehicle under these conditions is shown in Figure 10.44(h), with the coordinates of the rest position shown adjacent to the vehicle. For comparison, the same simulation was run with the right rear wheel free to rotate. The positions of the vehicle at 1-s intervals, corresponding to positions at the same time intervals for initial simulation with the right rear wheel locked, are shown in gray in Figure 10.44(h).

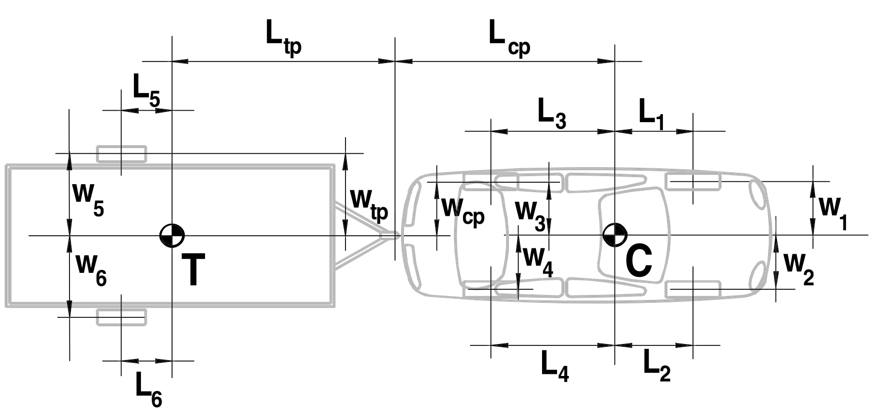


Figure 10.44 (e) Parameters for a vehicle with a trailer.

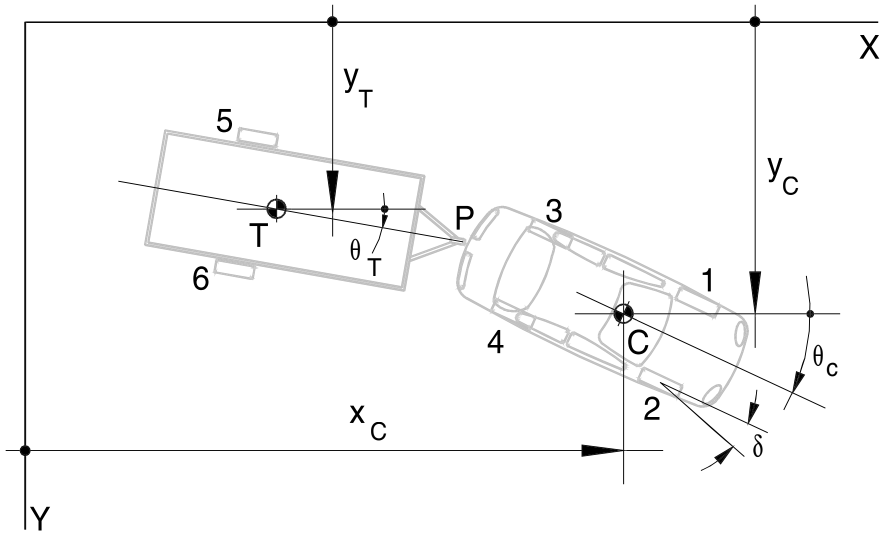
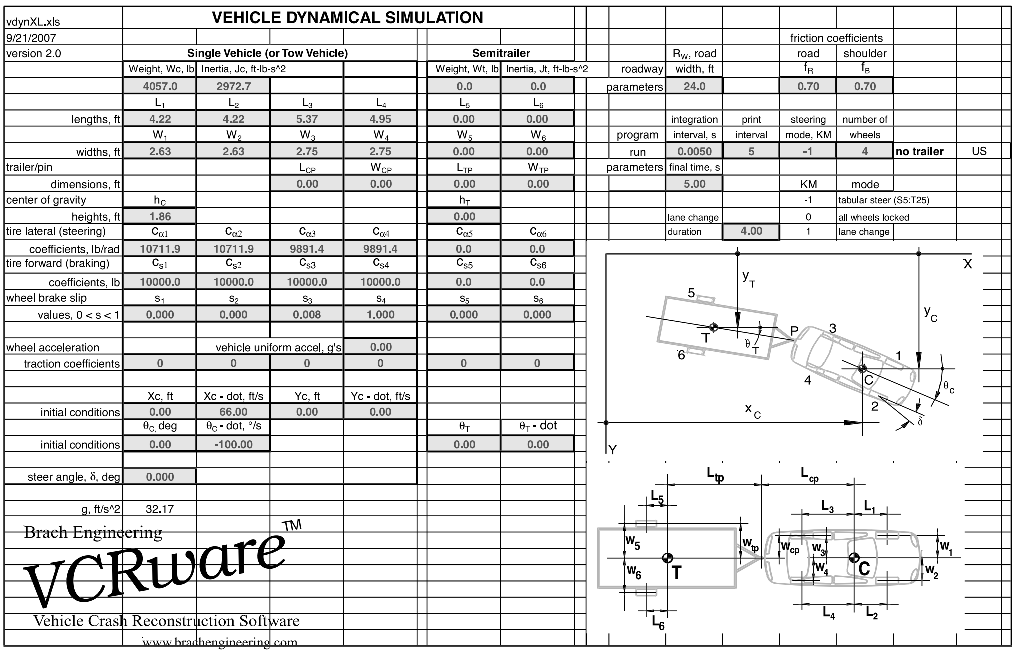


Figure 10.44 (f) Diagram showing the coordinates and variables associated with the vehicle dynamics simulation for a tow vehicle pulling a semi-trailer.



vdynXL.xls	VEHICLE DYNAMICAL SIMULATION					
9/21/2007						
version 2.0	Single Vehicle (or Tow Vehicle)				Semitrailer	
	Weight, Wc, lb	Inertia, Jc, ft-lb-s^2			Weight, Wt, lb	Inertia, Jt, ft-lb-s^2
	4057.0	2972.7			0.0	0.0
	L_1	L_2	L_3	L_4	L_5	L_6
lengths, ft	4.22	4.22	5.37	4.95	0.00	0.00
	W_1	W_2	W_3	W_4	W_5	W_6
widths, ft	2.63	2.63	2.75	2.75	0.00	0.00
trailer/pin			L_{CP}	W_{CP}	L_{TP}	W_{TP}
dimensions, ft			0.00	0.00	0.00	0.00
center of gravity	h_C				h_T	
heights, ft	1.86				0.00	
tire lateral (steering)	$C_{\alpha 1}$	$C_{\alpha 2}$	$C_{\alpha 3}$	$C_{\alpha 4}$	$C_{\alpha 5}$	$C_{\alpha 6}$
coefficients, lb/rad	10711.9	10711.9	9891.4	9891.4	0.0	0.0
tire forward (braking)	C_{s1}	C_{s2}	C_{s3}	C_{s4}	C_{s5}	C_{s6}
coefficients, lb	10000.0	10000.0	10000.0	10000.0	0.0	0.0
wheel brake slip	s_1	s_2	s_3	s_4	s_5	s_6
values, 0 < s < 1	0.000	0.000	0.008	1.000	0.000	0.000
wheel acceleration		vehicle uniform accel, g's		0.00		
traction coefficients	0	0	0	0	0	0
	Xc, ft	Xc - dot, ft/s	Yc, ft	Yc - dot, ft/s		
initial conditions	0.00	66.00	0.00	0.00		
	θ_C, deg	θ_C - dot, °/s			θ_T	θ_T - dot
initial conditions	0.00	-100.00			0.00	0.00
steer angle, δ, deg	0.000					
g, ft/s^2	32.17					

			friction coefficients			
	R_W, road		road	shoulder		
roadway	width, ft		f_R	f_B		
parameters	24.0		0.70	0.70		
	integration	print	steering	number of		
program	interval, s	interval	mode, KM	wheels		
run	0.0050	5	-1	4	**no trailer**	US
parameters	final time, s					
	5.00		KM	mode		
			-1	tabular steer (S5:T25)		
	lane change		0	all wheels locked		
	duration	4.00	1	lane change		

Figure 10.44 (g) Spreadsheet showing the data input for Case Study 2.

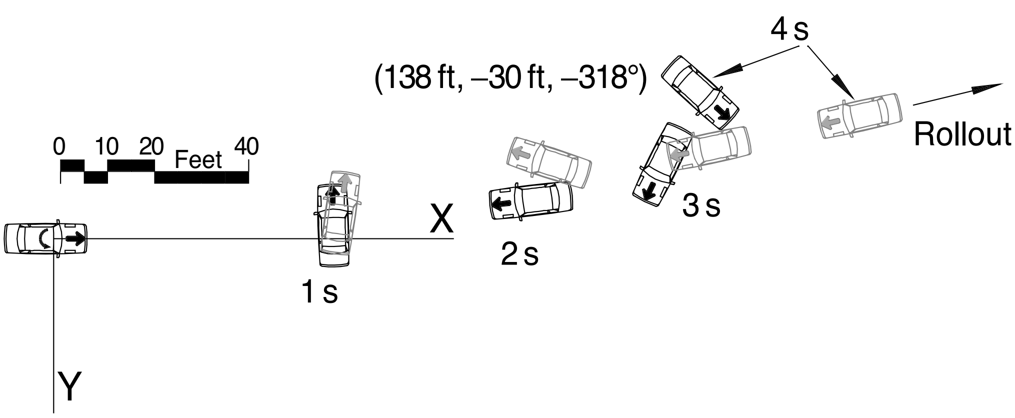


Figure 10.44 (h) Diagram showing the vehicle motion for simulation with the right rear wheel locked (black) and for the right rear wheel free to rotate (gray).

Problem 10.45

The Statistical Bounds of Accident Reconstruction: A Litigated Case History

By Albert G. Fonda
Fonda Engineering Associates

Abstract

This is a case study of one accident reconstruction from event analysis through the end of litigation.

This case study demonstrates the maturation of the art of accident reconstruction to a state such that the reconstructionist not only can recite a best opinion as to the event, as usual, but can quantitatively show why that opinion is "best," that is, numerical values now can be provided for the statistically likely bounds of variation in the results, given the available physical evidence.

Because the differential bounds found by one means of reconstruction are equally applicable to the central values found by any other means, the new treatment can be exercised as a "finishing touch" supplement to pre-existing central results found by any method of reconstruction.

When both experts are so prepared and make full disclosure, the litigants, during discussions in advance of trial, or, later, the judge or jury will be better informed in reaching a decision. When only one expert is so prepared, such a presentation will tend to prevail because of its completeness and candor. The opportunity for such a forensic advantage is not to be overlooked by the informed reconstructionist.

Introduction

Accident reconstruction, when it reasonably can be effective, relies on adequate measurement of the vehicles and their trajectories, whether made by local authorities, made by or for the eventual reconstructionist, or as published. But of course no measurement is exact, and recent clinical studies [1] have quantified the statistically likely bounds of error in such measurement. By classical but in this field novel statistical methods [2,6,7], the corresponding bounds of their consequences can be found. Such an offering will speak directly to the Daubert ruling, whereby the judge may properly find scientific testimony to be inadmissible for lack of recitation of its bounds of reliability.

Time-forward methods (simulations) [8] are unsuitable for such studies because they produce effects (damage and trajectories) given causes (approach conditions), so that the possible final-state measurements are not available to be modulated. This is why time-reversed methods [3-5,9,10], proceeding from effect to cause on the basis of conservation of momentum and energy, are required for any study of the effects of investigational measurement variability.

Monte Carlo methods, in which in consecutive solutions all the inputs are varied randomly on the basis of their known variability, followed by statistical characterization of the effects, are far less efficient than the method of finite difference analysis (FDA). As was reviewed most recently in Reference 7, in consecutive solutions, each input is varied once only to the extent of the statistically likely bounds of error in that measurement, followed by the probabilistic summation of consequences.

Given sufficient analysis time, FDA could be accomplished by repetitive use of any time-reversed algorithm, examples being the public-domain (NHTSA) program CRASH [9] and the present author's CRASHEX [3-5] program. This would require before each run the addition of a positive or negative perturbation to one of the input variables, after each run the finding and the storage of each change in every output of interest, repetition of these steps for each remaining input, and finally for each output the root-sum-squaring of the effects of all perturbations. However, the same repetitive procedures might be provided algorithmically, as they have been in CRASHEX. Eventual incorporation of similar routines into other time-reversed accident reconstruction programs is to be expected.

Case History

A representative accident reconstruction as used in actual litigation on behalf of the occupants of a left-turning vehicle will be shown for the fairly common case of impact by a rapidly oncoming vehicle.

Figure 10.45(a) shows the reported paths of the vehicles and their positions and yaw attitudes at impact and at rest, according to the applicable police report.

Vehicle 1 (Southbound) turned left across the path of Vehicle 2 (Northbound) after it had crested a hill 250 ft to the South. Positions of Vehicle 1 are shown before blocking and after clearing of the path of Vehicle 2.

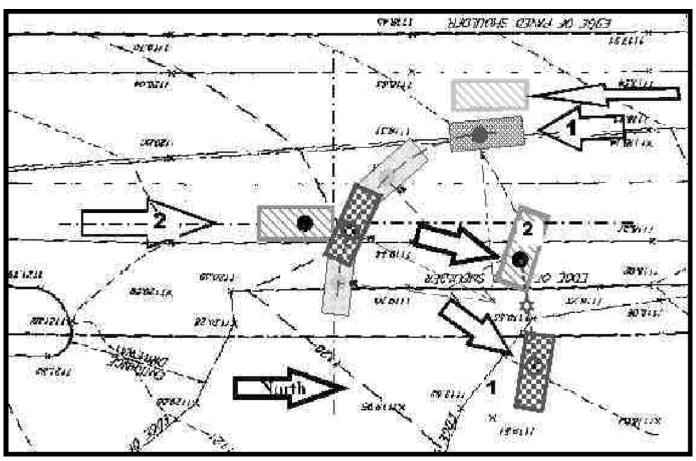


Figure 10.45 (a) Site map.

Although because of its anti-skid brakes, the tires of Vehicle 2 had left no visible marks, their noise had been reported by the driver of an adjacent Southbound vehicle [the unshaded outline in Figure 10.45(a)]. This was consistent with the report by Driver 2 of pre-impact hazard sighting and anti-skid pedal response.

The front of Vehicle 2 centrally impacted the right side of Vehicle 1, impelling it Northeast a measured 43' as it rotated clockwise to rest off-pavement facing West. Meanwhile, Vehicle 2 traveled Northeast a measured 34' 3" while rotating clockwise to rest facing slightly South of East with its front wheels on the East shoulder. The post-impact tire marks and rest positions were shown in police photographs.

Reconstruction

Such site data, plus vehicle data, provide the inputs needed for accident reconstruction by one means or another according to the laws of physics. In the subject case, CRASHEX produced a site diagram [Figure 10.45(b)], which matched the site map and the tire mark photographs.

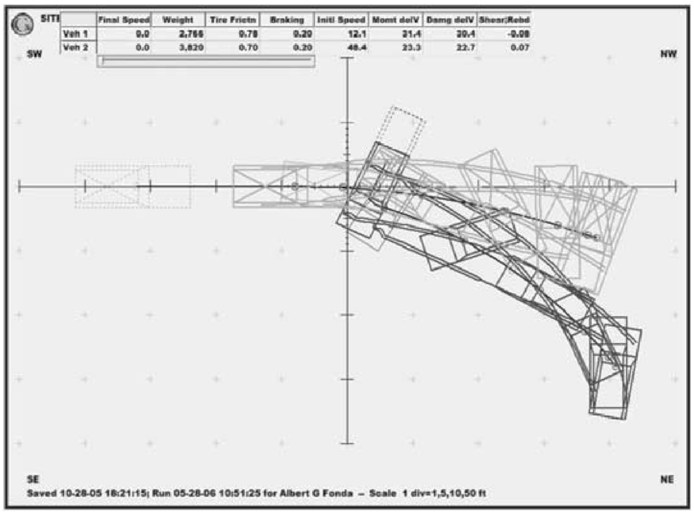


Figure 10.45 (b) Site plot.

Another diagram [left in Figure 10.45(c)] recapitulated the damage contour as numerically entered plus computed vectors for the impact force, accompanied (right) by a diagram of the vector solution being performed. In this vector diagram, OS and SM are the almost collinear momenta upon separation found from the trajectories to rest for Vehicles 1 and 2, respectively. From these have been found the almost perpendicular vectors OI and IM, the momenta upon impact in the respective directions of approach; the loss is due to calculation of tire forces [3]. Their vector difference IS≡-SI is the momentum exchanged between the vehicles during impact. Through energy considerations, the program provided a second (here indistinguishable) vector magnitude IS≡-SI.

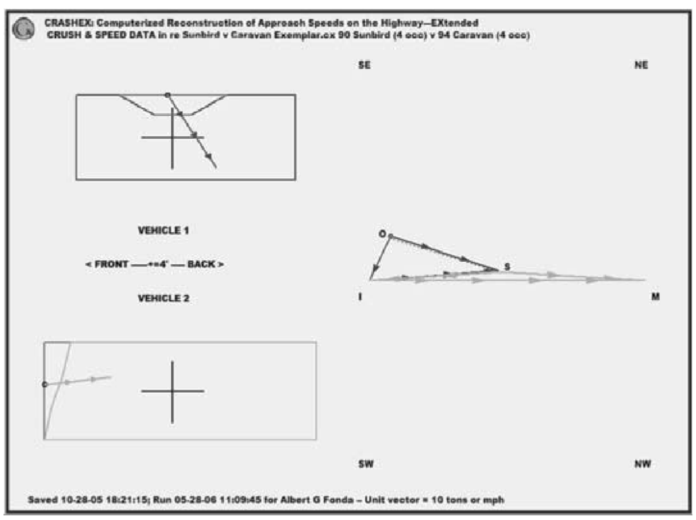


Figure 10.45 (c) Vector plot.

The program also issued a numerical tabulation of these and other findings—68 outputs in all.

Results

In the subject case, as in many numerical outputs of primary interest, are the two approach (impact) speeds, which in this instance are

- 12.1 mph for Vehicle 1 and
- 48.4 mph for Vehicle 2

as seen also in the supplemental table, which is an inset in Figure 10.45(b).

Given the approach speeds, any reconstruction may be extrapolated back to the time when the hazard might have been recognized and avoided by a participant. Evidently here, as in most left-turn impacts, each driver took the other by surprise; however, at turn initiation, a hidden or sufficiently distant oncoming vehicle would have presented no evident hazard. The apparent responsibility for avoidance thus depends critically on the sight distance to and the reconstructed speed of the oncoming vehicle.

It is obvious that Vehicle 2, impacting at a speed most likely to have been 48.4 mph following heavy and prolonged avoidance braking, must have begun to brake when moving at well over the applicable 55 mph speed limit. The degree of excess remains to be quantified.

The adjacent Southbound witness had heard tire squeal, turned her head to the left about 90°, and saw the impact occur. This indicated, as a tentative value adequate for settlement purposes, a witness response time of 1 s.

As is shown in Figure 10.45(d), that assumption was combined in a spread sheet with available driving-simulator operator time histories and the probable tire-road downhill deceleration of Vehicle 2. Given the anti-skid braking, the friction coefficient was taken to be 0.85. The figure shows that the braking time history from hazard detection through pedal response to brake application, for a rise in 1 s to a steady-state value of 0.823 (downhill) at the start of tire noise, continued while the witness responded and saw the impact occur.

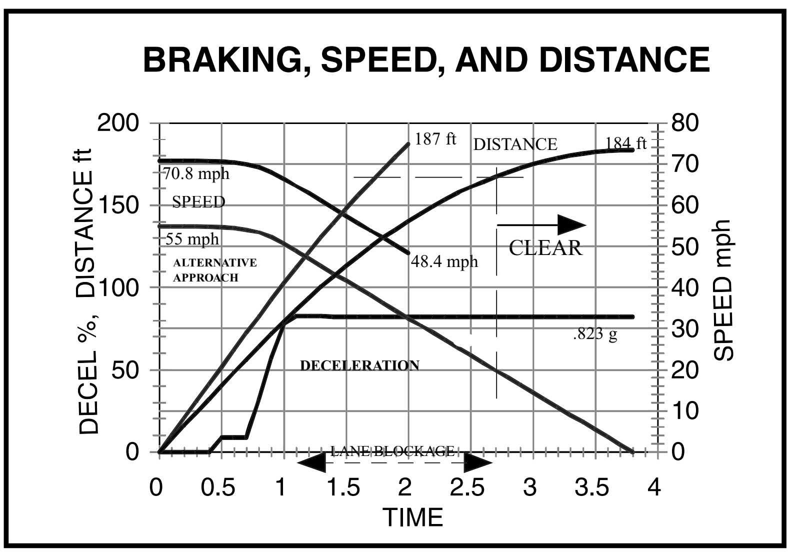


Figure 10.45 (d) Approach scenario—base case.

By numerical integration from the probable final speed of 48.4 mph, the probable initial speed was 70.8 mph, with travel distance of 187 ft from hazard recognition to impact. The speed before braking thus was 15.8 mph in excess of the 55 mph legal limit. The point of recognition was 250 − 187 = 63 ft and 0.6 s past the crest of the hill—enough time for Driver 2 to respond.

As to avoidance, but for the speed violation the initial speed would have been 55 mph. Given the same braking, the time and distance to stop would have been 3.8 s and 184 ft. Because this is 3 ft short of the path of Vehicle 1, regardless of the timing of Vehicle 1, no impact could have occurred.

Furthermore, by the time Vehicle 1 had cleared the Northbound lane (broken vertical line), Vehicle 2 would have traveled only 167 ft, arriving at 20 mph at 187 − 167 = 20 ft from the departing right rear of Vehicle 1. Impact thus would have been avoided, and each vehicle would have gone its way unaffected.

Ordinarily, this would have been the end point of the reconstruction, leaving the reconstructionist with nothing substantive to say about the quality or reliability of such a "best estimate." However, as decisive as the above 20 ft of clearance might seem, it would be fairly questioned. The presenting expert, much less opposing counsel, would be concerned as to its reliability because no measurement can be made exactly. How reliable, given the established evidence, is the finding?

This is exactly the question that FDA can quantitatively answer. Although credible evidence that is presented as a hypothetical is not to be questioned, every measurement credibly may have been as much in error as the high or the low extreme of the range experimentally shown to be likely under similar circumstances. All of these deviations can be assumed to have occurred in a random manner, and their equally likely combined effect can be found.

Independent groups have long evaluated and published various tables of vehicle weights and measures and experimental values of tire-road friction, sufficient for recitation of inferred means and standard deviations by category. Absent, however, was corresponding information for the performance of field measurement tasks through the use of various procedures and equipment. In 2000, by enlisting the cooperation of attendees at a conference of accident reconstructionists, through the efforts of initiator William Wright and others [1], this paucity of information was remedied. Now, the business of "reliability reconstruction" could proceed [2], finally enabling, in CRASHEX, the effective usage of algorithms that had been written more than a decade before.

Finite Difference Analysis

Finite difference analysis commences after such a central or best-case reconstruction has been achieved. In the first stage of FDA, user-specified changes, all statistically equally likely, are applied in turn to each of the inputs to the established base case. In the second stage, those results are statistically summed.

In CRASHEX FDA, each entry made into an otherwise blank input array is treated as a change of input that immediately produces a full array of differential outputs, tabulated in the same format as the outputs that are being perturbed. Although each input could be a unit quantity (e.g., 1 meter or 1 foot or 1 pound) producing an array of sensitivities or transfer coefficients, when each such input is set to equal the span of probable uncertainty or deviation in the associated measurement, each such output array lists the consequent and equally probable deviation of each output. Presumption of 2 SD deviations of measurement is the norm, as such inputs will include 95.44% of the probable input and (hence) output deviations, excluding at each end of the spectrum only 2.28% (1 part in 44) of the possible span.

For the exemplar case, by using as inputs twice the "Medium" (in most instances) standard deviations of measurement found [2,6,7] in juried and other studies, 43 input differences were used consecutively in 43 runs of the FDA routine. For each run, a single printout listed the change in some one of the inputs and the consequent changes in all 68 outputs. These 43 sheets could then be sorted to rank the inputs in terms of their effect on any one output of interest. This procedure identifies for each output "the significant few" influences—usually only the top three or four, as will be demonstrated—among "the trivial many" [2].

In the second stage of FDA, for each output of interest (or for all of them, in CRASHEX), the root of the sum of the squares of all previously calculated output differences is found. Given 2 SD input deviations, this sum is the 2 SD reliability of each output, the "95% reliability side-band" values that include all but 1 part in 44 of the effects at each end of all probable deviations of measurement.

By basic statistical theory, this is the probable combined effect of the random occurrence of all input deviations. Vectorially, it is a progressive summation of successive normals to each preceding sum. Such summation avoids the excessively conservative linear summation of the absolute values of all effects, yet preserves a due effect for every influence.

For the exemplar case, the speed of Vehicle 1 has a span of uncertainty of 5.5 mph. Although this is 45% of its 12.1 mph central value, this speed is of little importance in the present dispute.

For the 48.4 mph impact speed of Vehicle 2, the total and its four most influential contributors were [see Figure 10.45(e)] as follows:

- 3.9 mph due to all 43 factors combined, including
  - 2.3 mph due to ±0.14 (Medium) changes in the tire-road friction of Vehicle 2
  - 1.9 mph due to ±0.14 (Medium) changes in the tire-road friction of Vehicle 1
  - 1.3 mph due to ±10 (Very High) changes in the approach direction of Vehicle 1
  - 1.0 mph due to ±130 pound (Medium) changes in the weight of Vehicle 1

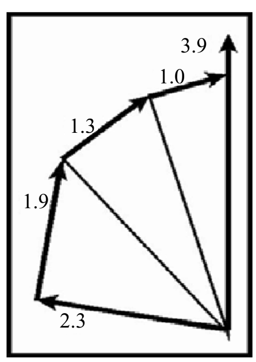


Figure 10.45 (e) Vector Sum

These four largest influences collectively account for 3.4 mph of the 3.9 mph total; the 39 others, only the last 0.5 mph. Thus, it is unnecessary for the expert (and it will be futile for the opposition) to be concerned with the exact variability of any input other than these four; no reasonable changes in them could be of much consequence. FDA confers immunity from further reasonable doubt, given the evidence.

Changes larger than any considered were held to be unlikely by, in most instances, standards variously published or specified independently of the present case. A ±10° (Very High) allowance for variation in the direction of approach of Vehicle 1 was assigned because that path could be deduced only from its end tangents and the point of impact. Even so, its variability was less important than that of the post-impact tire-road friction, which, given published generic data, was allowed to vary from 0.56 to 0.84 for Vehicle 1 and from 0.64 to 0.92 for Vehicle 2.

Because controlling transfer functions depend only on the physics (or the vector diagram) of the event, the differential bounds found by one means of reconstruction are equally applicable to the central values found by any other means. This permits superposition on a previous reconstruction of the probability side-bands found with subsequent use of an FDA-facilitated treatment of the same case.

As to the approach scenario, if for settlement purposes the previous estimate of 2 s of approach time following first alert is retained, along with the same friction variability of ±0.14, for the case of improbably high friction of 0.963 before impact at 52.3 mph, with 1 chance in 44, Vehicle 2 was found to have traveled

- 205 ft from an initial speed of
  - 77.6 mph, which was
    - 22.6 mph over the 55 mph limit

Although indicative of very prompt observance of Vehicle 1 by Driver 2 while sighting over the crest of the hill, this scenario cannot be ruled out. However, this is not the pivotal case. It is in other respects equally likely that, for the case of improbably low deceleration of 0.683 before impact at 44.5 mph, Vehicle 2 traveled

- 168 ft from an initial speed of
  - 63.6 mph, which was only
    - 8.6 mph over the 55 mph limit

As to the alternative scenario, had the pre-response approach speed of Vehicle 2 been 55 mph, then for the higher-friction case, a full stop in

- 3.4 s and
- 169 ft was possible, or
- 36 ft short of impact. But
- 0.7 s before that, at
- 15 mph and with
- 44 ft of clearance
- 2.7 s the lane was clear, and braking could have been terminated at

and no collision would have ensued. For the lower-friction case (with ▾ symbols), a full stop in

- 4.2 s and
- 206 ft would have been possible if unimpeded. But
- 1.65 s before that, at
- 2.55 s, at
- 168 ft, the right front bumper of Vehicle 2 would have impacted at
- 27 mph the last
- 2 ft of Vehicle 1

with collision ensuing. So the more critical case occurs for the lower tire-road friction, and within the reasonable bounds of evidence, total avoidance of impact cannot be guaranteed. This is the adverse "what-if" contingency for which the reconstructionist should have been prepared, and about which he or she would properly have been cross-examined.

But note that Vehicle 1 would have vacated the path of Vehicle 2 had it not arrived by the time t = 2.7 s. By interpolation, this occurs for 0.714 deceleration, after 172 ft of travel, with arrival at 24 mph. With 16 chances in 17, the right front corner of Vehicle 2 would arrive too late to contact the departing right rear corner of Vehicle 1 [see Figure 10.45(a)].

Even in the remaining 1 chance in 17, both the intervehicular impact and the now-distal occupant-vehicle impact would have been drastically reduced. But what is very much more likely is that but for the speed violation, neither the vehicles nor their occupants would have been impacted at all.

On occasion, the merely preliminary and verbal recital of such a significant finding by the client's attorney would be sufficient for settlement. Because this was so in the subject case, to the plaintiff's benefit, we cannot recite the much more extensive preparation that would have followed in a more protracted litigation. Well-qualified experts would have been needed for independent assessment of the human factor values and the injury reduction, and demonstrative exhibits would have included detailed simulations and animations of the event for the various scenarios reviewed here.

Conclusions

For an exemplar case, the use of a time-reversed collision reconstruction program incorporating an automated FDA routine has accurately established both a most likely case and the credible departures from that case due to uncertainties of measurement.

Ordinarily, only the central case could have been presented. Although that would have indicated impact avoidance but for a speed violation, the expert could have provided no analytic rebuttal to the suggestion of a contrary finding within the range of known evidence.

Absent reliability side-bands, the susceptibility of any bare finding to inevitable but unknown errors of measurement can never be denied, nor can finders of fact be spared the necessity of sheer speculation in this regard. Neither can any forensic expert otherwise offer more than a supposedly educated guess, a sheer expression of personal confidence lacking in analytic foundation, as to the magnitude of such susceptibility. Finite difference analysis provides the practical remedy to this shortcoming and contributes critically to the forensic utility of accident reconstruction.

AutoLibrary

Whatever the treatment of choice and whatever its constraints, this approach brings to accident reconstruction in general a salutary level of sophistication. The practitioner not only can present as usual the treatment and the reconstruction that in his or her opinion are most reliable, but now can, through brief further use of any FDA-enabled time-reversed reconstruction, recite also the probable boundaries of that reconstruction, given the practical limitations of investigative accuracy. All "what-if" questions issuing from the inherent uncertainty of physical measurement thus can be answered before they are asked.

Contact

Mr. Fonda can be reached at agfonda@gmail.com. Together with his profile, the accident reconstruction and finite difference analysis program CRASHEX is accessible at www.crashex.com in a form suitable for abbreviated usage in time-share style at a minimal fee with simple pass-through of per-case expense to the end user.

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Appendix A

Vector Algebra

Problem A.1

Find the length of the resultant of the following vectors.

$$
\begin{array}{c}
3i + 4j - 5k \\
7i + 2j - 5k \\
-12i - 11j + 2k
\end{array}
$$

Solution to Problem A.1

The resultant is produced by adding the vectors

$$
\begin{array}{r}
3i + 4j - 5k \\
7i + 2j - 5k \\
\underline{-12i - 11j + 2k} \\
-2i - 5j + 8k
\end{array}
$$

The length of the resultant vector is

$$L = \sqrt{(-2)^2 + (-5)^2 + (8)^2} = \sqrt{93} = 9.644$$

Problem A.2

Find the unit vector associated with the vector.

$$18i + 6j + 25k$$

Solution to Problem A.2

The unit vector of a particular vector is the vector itself divided by its length.

$$\text{Unit vector} = \frac{18i + 6j + 25k}{\sqrt{(18)^2 + (6)^2 + (25)^2}} = \frac{18i + 6j + 25k}{\sqrt{985}}$$

$$= \frac{18i + 6j + 25k}{31.385} = 0.574i + 0.191j + 0.797k$$

Problem A.3

What is the angle between the two vectors a and b?

$$a = 7i + 10j + 6k$$
$$b = 24i - 8j + 5k$$

Solution to Problem A.3

There are two ways to calculate the dot product of vectors. Apply both and set their results equal to each another.

$$a \cdot b = |a|\,|b| \cos\theta = \sqrt{(7)^2 + (10)^2 + (6)^2} \times \sqrt{(24)^2 + (-8)^2 + (5)^2} \cos\theta$$

$$= (13.6)(25.79)\cos\theta$$

$$a \cdot b = a_x b_x + a_y b_y + a_z c_z = (7)(24) + (10)(-8) + (6)(5) = 783$$

$$(13.6)(25.79)\cos\theta = 783$$

$$\cos\theta = 2.232$$

$$\theta = \cos^{-1}[2.232] \text{ (in degrees).}$$

Problem A.4

Given $a = 2i + 4j + 6k$ and $b = -2i + j - 5k$, find $a \cdot b$.

Solution to Problem A.4

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = 2(-2) + 4(1) + 6(-5) = -30$$

Problem A.5

Given the vectors

$$a = 2i + 3j + 5k$$
$$b = -2i + j - 7k$$

find the dot product, $a \cdot b$ of the vectors.

Solution to Problem A.5

Multiply the corresponding components of the vectors and add the resulting products.

$$a \cdot b = a_x b_x + a_y b_y + a_z c_z = (2)(-2) + (3)(1) + 5(-7) = -4 + 3 - 35 = -36$$

Problem A.6

Find the cross product, a × b of the vectors

$$a = 2i + 4j + 5k$$
$$b = -2i + 3j - 4k$$

Solution to Problem A.6

Find the determinant of the 3 × 3 matrix whose rows are the unit vectors and the coefficient of vectors a and b.

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i(a_y b_z - b_y a_z) - j(a_x b_z - b_x a_z) + k(a_x b_y - b_x a_y)$$

$$= \begin{vmatrix} i & j & k \\ 2 & 4 & 5 \\ -2 & 3 & -4 \end{vmatrix} = i\begin{vmatrix} 4 & 5 \\ 3 & -4 \end{vmatrix} - j\begin{vmatrix} 2 & 5 \\ -2 & -4 \end{vmatrix} + k\begin{vmatrix} 2 & 4 \\ -2 & 3 \end{vmatrix}$$

$$= i(-16 - 15) - j(-8 + 10) + k(6 + 8)$$
$$= i(-31) -2j + 14k$$

Problem A.7

Find the cross product, a × b of the vectors

$$a = 2i + 5j + 8k$$
$$b = -2i + j - 5k$$

Solution to Problem A.7

$$\begin{vmatrix} i & j & k \\ 2 & 5 & 8 \\ -2 & 1 & -5 \end{vmatrix}$$

$$a \times b = i\begin{vmatrix} 5 & 8 \\ 1 & -5 \end{vmatrix} - j\begin{vmatrix} 2 & 8 \\ -2 & -5 \end{vmatrix} + k\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}$$

$$= i(-25 - 8) - j(-10 + 16) + k(2 + 10)$$
$$= -33i - 6j + 12k$$

Problem A.8

If a = (–3, 1, 6) and b = (0, 5, –4), calculate a × b, b × a and a × a.

Solution to Problem A.8

$$a \times b = \begin{vmatrix} i & j & k \\ -3 & 1 & 6 \\ 0 & 5 & -4 \end{vmatrix} = i\begin{vmatrix} 1 & 6 \\ 5 & -4 \end{vmatrix} - j\begin{vmatrix} -3 & 6 \\ 0 & -4 \end{vmatrix} + k\begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix}$$

$$= i[-4 - 30] - j[12 - 0] + k[-15 - 0] = -34i - 12j + 15k$$

$$b \times a = \begin{vmatrix} i & j & k \\ 0 & +5 & -4 \\ -3 & 1 & 6 \end{vmatrix} = i\begin{vmatrix} 5 & -4 \\ 1 & -6 \end{vmatrix} - j\begin{vmatrix} 0 & -4 \\ -3 & 6 \end{vmatrix} + k\begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix}$$

$$= i[30 + 4] - j[0 - 12] + k[0 + 15] = 34i + 12j - 15k$$

$$a \times a = \begin{vmatrix} i & j & k \\ -3 & 1 & 6 \\ -3 & 1 & 6 \end{vmatrix} = i\begin{vmatrix} 1 & 6 \\ 1 & 6 \end{vmatrix} - j\begin{vmatrix} -3 & 6 \\ -3 & 6 \end{vmatrix} + k\begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix}$$

$$= i[6 - 6] - j[-18 - 18] + k[-2 + 3] = 0i - 0j + 0k$$

Problem A.9

Find the cross product, $a \times b$ of the vectors a and b.

$$a = i + 5j + 6k$$
$$b = 8i + 5j + 5k$$

Solution to Problem A.9

The cross product of two vectors is the determinant of a third-order matrix, shown as follows.

$$a \times b = \begin{bmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} i & j & k \\ 1 & 5 & 6 \\ 8 & 5 & 5 \end{bmatrix} = i\begin{vmatrix} 5 & 6 \\ 5 & 5 \end{vmatrix} - j\begin{vmatrix} 1 & 6 \\ 8 & 5 \end{vmatrix} + k\begin{vmatrix} 1 & 5 \\ 8 & 5 \end{vmatrix}$$

$$= i[25 - 30] - j[5 - 48] + k[5 - 40] = -5i + 43j - 35k$$

Problem A.10

If $a = 2i - 3j - 5k$ and $b = i + 2j + 3k$, find

a) $a \times b$
b) $b \times a$
c) $(a + b) \times (a - b)$

Solution to Problem A.10

$$a = 2i - 3j - 5k$$
$$b = i + 2j + 3k$$

a) $a \times b = (2i - 3j - 5k) \times (i + 2j + 3k)$

$$= \begin{bmatrix} i & j & k \\ 2 & -3 & -5 \\ 1 & 2 & 3 \end{bmatrix} = i\begin{vmatrix} -3 & -5 \\ 2 & 3 \end{vmatrix} - j\begin{vmatrix} 2 & -5 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$= i[(-3)(3) - (2)(-5)] - j[(2)(3) - (1)(-5)] + k[(2)(2) - (1)(-3)]$$
$$= i[-9 + 10] - j[6 + 5] + k[4 + 3] = i + 11j + 7k$$

b) $b \times a = (i + 2j + k) \times (2i - 3j - k)$

$$= \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -3 & -5 \end{bmatrix} = i\begin{vmatrix} 2 & 3 \\ -3 & -5 \end{vmatrix} - j\begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$$

$$= i[-10 + 9] - j[-5 - 6] + k[-3 -4] = -i + 11j - 7k$$

c) $(a + b) \times (a - b)$

$a = 2i - 3j - 5k$
$b = i + 2j + 3k$
$a + b = (2i - 3j - 5k) + (i + 2j + 3k) = 3i - j - 2k$
$a - b = (2i - 3j - k) - (i + 2j + k) = i - 5j - 8k$
$(a + b) \times (a - b) = (3i - j - 2k) \times (i - 5j - 8k)$

$$= \begin{bmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & -5 & -8 \end{bmatrix} = i\begin{vmatrix} 3 & -2 \\ 1 & -8 \end{vmatrix} - j\begin{vmatrix} 3 & -2 \\ 1 & -8 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 1 & -5 \end{vmatrix}$$

$$= i[-24 + 2] - j[-24 + 2] + k[-15 + 1] = -22i + 22j - 14k$$

Problem A.11

If $a = 5i - j + 2k$, $b = i + 3j + k$ and $c = i + 3j + 2k$, find

a) $(a \times b) \times c$

b) $a \times (b \times c)$

Solution to Problem A.11

$$a = 5i - j + 2k$$
$$b = i + 3j + k$$
$$c = i + 3j + 2k$$

a) $a \times b = (5i - j + 2k) \times (i + 3j + k)$

$$= \begin{bmatrix} i & j & k \\ 5 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix} = i\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} - j\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= i[-1 - 6] - j[5 - 2] + k[15 + 1] = -7i - 3j + 16k.$$

$(a + b) \times c = (-7i - 3j + 16k) \times (i + 3j + 2k)$

$$= \begin{bmatrix} i & j & k \\ -7 & -3 & 16 \\ 1 & 3 & 2 \end{bmatrix} = i\begin{vmatrix} -3 & 16 \\ 3 & 2 \end{vmatrix} - j\begin{vmatrix} -7 & 16 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} -7 & -3 \\ 1 & 3 \end{vmatrix}$$

$$= i[-6 - 48] - j[-14 - 16] + k[-21 + 3] = -54i + 30j - 21k.$$

b) $b \times c = (i + 3j + k) \times (i + 3j + 2k)$

$$= \begin{bmatrix} i & j & k \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} = i\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$$

$$= i[6 - 3] - j[2 - 1] + k[3 - 3] = 2i - j + 0k.$$

$a \times (b \times c) = (5i - j + 2k) \times (3i - j + 0k)$

$$= \begin{bmatrix} i & j & k \\ 5 & -1 & 2 \\ 3 & -1 & 0 \end{bmatrix} = i\begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} - j\begin{vmatrix} 5 & 2 \\ 3 & 0 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ 3 & -1 \end{vmatrix}$$

$$= i[-1 + 2] - j[0 - 6] + k[-5 + 3] = i - 6j - 2k.$$

From (a) and (b), clearly $\quad (a \times b) \times c \neq a \times (b \times c)$

Problem A.12

For the three vectors a, b, and c, what is the product $a \cdot (b \times c)$?

$$a = 6i + 7j + 10k$$
$$b = i + 2j + 5k$$
$$c = 3i + 6j + 5k$$

Solution to Problem A.12

$$b \times c = \begin{bmatrix} i & j & k \\ 1 & 2 & 5 \\ 3 & 6 & 5 \end{bmatrix} = i\begin{vmatrix} 5 & 5 \\ 6 & 5 \end{vmatrix} - 1\begin{vmatrix} j & k \\ 6 & 5 \end{vmatrix} + 3\begin{vmatrix} j & k \\ 2 & 5 \end{vmatrix}$$

$$= i(10 - 30) - 1(j5 - 6k) + 3(j5 + 2k) = -20i + 10j.$$

Now calculate the dot product

$$a \cdot (b \times c) = (6)(-20) + (7)(10) + (10)(10) = -120 + 70 = 50$$

Problem A.13

Given the vectors a, b, and c, what is the value of $(a + b) \cdot (b + c)$?

$$a = 6i + 3j + 2k$$
$$b = 4i + 3j + 4k$$
$$c = 6i + 7j + 10k$$

Solution to Problem A.13

Sum the like components of the vectors being added.

$$\begin{array}{rr} a + b = & 6i + 3j + 2k \\ & \underline{4i + 3j + 4k} \\ & 10i + 6j + 6k \end{array}$$

$$\begin{array}{rr} b + c = & 4i + 3j + 4k \\ & \underline{6i + 7j + 10k} \\ & 10i + 10j + 14k \end{array}$$

The dot product is the sum of the products of the like components.

$$(a + b) \cdot (b + c) = (10)(10) + (6)(10) + (6)(14) = 244J.$$

Problem A.14

Find the volume of the parallelepiped $[(a \times b) \cdot c]$ determined by the vectors

$$a = (3, -2, 0),\ b = (2, 2, 2) \text{ and } c = (-1, -1, 5)$$

Solution to Problem A.14

$$a \times b = \begin{bmatrix} i & j & k \\ 3 & -2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = i\begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix} + k\begin{vmatrix} 3 & -2 \\ 2 & 2 \end{vmatrix}$$

$$= i[-4 - 0] - j[6 - 0] + k[6 + 4] = -4i - 6j + 10k.$$

Volume of the parallelepiped is

$$|\,a \times b \cdot c| = |(-4, -6, 10) \cdot (-1, -1, 5)| = 60$$

Problem A.15

Determine the volume of a parallelepiped with sides represented by the zero-based vectors a, b, and c.

$$\begin{aligned} a &= 2i - 2j + 3k \\ b &= 4i + 2j + 5k \\ c &= i + 5j + 6k \end{aligned}$$

Solution to Problem A.15

$$\text{Volume of the parallelepiped} = \begin{vmatrix} 2 & -2 & 3 \\ 4 & 2 & 5 \\ 1 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 5 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} -2 & 3 \\ 5 & 6 \end{vmatrix} + 1\begin{vmatrix} -2 & 3 \\ 2 & 5 \end{vmatrix}$$

$$\begin{aligned} &= 2[12 - 25] - 4[-12 - 15] + 1[-10 - 6] \\ &= 2(-13) - 4(-27) + 1(-16) \\ &= -26 + 108 - 16 = 108 - 42 = 66. \end{aligned}$$

Appendix B

Matrix Analysis

Example Problems and Solutions

Example B.1

Given $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 6 & 5 \end{bmatrix}$, find $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$.

Solution:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+4 & 3+1 & 2+3 \\ 5+2 & 4+6 & 6+5 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 5 \\ 7 & 10 & 11 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-4 & 3-1 & 2-3 \\ 5-2 & 4-6 & 6-5 \end{bmatrix} = \begin{bmatrix} -3 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

MATLAB Solution:

```
>> A= [1 3 2; 5 4 6];
>> B= [4 1 3; 2 6 5];
>> A+B
ans =
     5     4     5
     7    10    11
>> A-B
ans =
    -3     2    -1
     3    -2     1
```

Example B.1

Consider the matrix A is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 6 & 4 \\ 2 & 4 & 7 \end{bmatrix}$$

Then adjoint of **A** is given by

$$\text{adj } \mathbf{A} = \begin{bmatrix} \begin{vmatrix} 6 & 4 \\ 4 & 7 \end{vmatrix} & -\begin{vmatrix} -1 & 4 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} -1 & 6 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & -3 \\ 6 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & -3 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 6 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 26 & 15 & -16 \\ -19 & 6 & 2 \\ 22 & 3 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 26 & -19 & 22 \\ 15 & 6 & 3 \\ -16 & 2 & 1 \end{bmatrix}$$

Example B.2

Find the matrix difference **A – B** of matrices **A** and **B**.

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 2 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 5 & 2 & -6 \\ -5 & 4 & 2 \end{bmatrix}$$

Solution:

The entries in the difference matrix are the difference of the corresponding entries in the original two matrices.

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 3-5 & 4-2 & 7-(-6) \\ 5-(-5) & 2-4 & -2-2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 13 \\ 10 & -2 & -4 \end{bmatrix}$$

MATLAB Solution:

```
>> A= [3 4 7; 5 2 -2];
>> B= [5 2 -6;-5 4 2];
>> A-B

ans =

    -2     2    13
    10    -2    -4
```

Example B.2

Consider a 3×3 square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 6 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\det \mathbf{A} = 1(2-18) - 3(3-12) + 4(9-4) = 31 \neq 0$$

Hence, the rank of matrix **A** is 3.

Example B.3

Given $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$, show that $\mathbf{AB} \neq \mathbf{BA}$.

Solution:

$$\mathbf{AB} = \begin{bmatrix} 2(1)+1(0) & 1(1)+0(0) & 5(1)+1(0) \\ 2(2)+1(1) & 1(2)+0(1) & 5(2)=1(1) \\ 2(4)+1(3) & 1(4)+0(3) & 5(4)+1(3) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 5 \\ 5 & 2 & 10 \\ 11 & 5 & 23 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 16 \\ 5 & 3 \end{bmatrix}$$

Clearly, $\mathbf{AB} \neq \mathbf{BA}$.

MATLAB Solution:

```
>> A=[3 4 7;5 2 -2];
>> B=[5 2 -6;-5 4 2];
>> A-B

ans =

    -2     2     13
    10    -2     -4

>> A=[1 0;2 1;4 3];
>> B=[2 1 5;1 0 1];
>> A*B

ans =

     2      1      5
     5      2     11
    11      4     23

>> B*A
```

```
ans =

    24    16
     5     3
Clearly, AB ≠ BA.
```

Example B.3

Determine the rank of the given matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Solution:

Because **A** is of the order of 3×4, the maximum possible value of the rank should be 3. This is true when at least one square submatrix of order 3 has non-zero determinant. Otherwise, the rank will be at most 2 and it must be investigated for square matrices of order 2.

There can be four square submatrices of order 3 in the given matrix, and these are as follows:

Sub matrix	Its determinant
$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$	0
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$	0
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$	0
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$	0

Hence, it is required to investigate submatrices of order 2.

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	1

Hence, a non-zero determinant is arrived at, and so we can conclude that the rank of $\mathbf{A} = 2$.

Example B.4

Find the matrix product **AB** of matrices **A** and **B**.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Solution:

Multiply the elements of each row in matrix **A** by the elements of the corresponding column in matrix **B**.

$$\mathbf{AB} = \begin{bmatrix} 2 \times 4 + 1 \times 2 & 2 \times 3 + 1 \times 1 \\ 1 \times 4 + 8 \times 2 & 1 \times 3 + 8 \times 1 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 20 & 11 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[2 1;1 8];
>> B=[4 3;2 1];
>> A*B

ans =

    10     7
    20    11
```

Example B.4

Consider two matrices **A** and **B** given by:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ -5 & 7 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 7 & 4 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 7 & 9 \\ 2 & 11 \end{bmatrix}; \quad \mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & -3 \\ -12 & 3 \end{bmatrix}$$

Example B.5

Find the matrix product **AB** of matrices **A** and **B**.

$$\mathbf{A} = [7 \quad 2 \quad 3] \qquad \mathbf{B} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

Solution:

$$\mathbf{AB} = [7 \times 2 + 2 \times (-3) + 3 \times 4] = 20.$$

MATLAB Solution:

```
>> A= [7 2 3];
>> B= [2; -3; 4];
>> A*B

ans =

    20
```

Example B.5

Consider two matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 6 & -3 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} -2 & 1 \\ 5 & 7 \end{bmatrix}$$

Then $\mathbf{AB} = \begin{bmatrix} 2(-2)+3(5) & 2(1)+3(7) \\ 6(-2)+(-3)5 & 6(1)+(-3)7 \end{bmatrix} = \begin{bmatrix} 11 & 23 \\ -27 & -15 \end{bmatrix}$

and $\mathbf{BA} = \begin{bmatrix} (-2)2+1(6) & (-2)3+1(-3) \\ (5)2+(7)6 & 5(3)+7(-3) \end{bmatrix} = \begin{bmatrix} 2 & -9 \\ 52 & -6 \end{bmatrix}$

Hence, $\mathbf{AB} \neq \mathbf{BA}$.

Example B.6

Given

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 8 & 0 \\ 2 & 7 \\ -1 & 4 \end{bmatrix}$$

Find $\mathbf{C} = \mathbf{AB}$.

Solution:

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 6 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 2 & 7 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 8+3\times 2+4\times(-1) & 2\times 0+3\times 7+4\times 4 \\ 1\times 8+(-5)\times 2+6\times(-1) & 1\times 0+(-5)\times 7+6\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 37 \\ -8 & -11 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[2 3 4;1 -5 6];
>> B=[8 0;2 7; -1 4];
>> C=A*B

C =

    18    37
    -8   -11
```

Example B.6

Given the two 2×2 matrices:

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

show that det **AB** = (det **A**) (det **B**).

Solution:

$$\det \mathbf{A} = 5 - 0 = 5$$
$$\det \mathbf{B} = 12 - 2 = 10$$

$$\mathbf{AB} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 9 & 7 \end{bmatrix}$$

$$\det \mathbf{AB} = 20(7) - 9(10) = 50$$
$$(\det \mathbf{A})(\det \mathbf{B}) = (5)(10) = 50 = \det \mathbf{AB}$$

Example B.7

Determine the following matrix product

$$\begin{bmatrix} 5 & 2 & 4 \\ 4 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \\ -5 & 4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 5 & 2 & 4 \\ 4 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \\ -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 3 + 2 \times 1 + 4(-5) & 5 \times 2 + 2 \times 7 + 4 \times 4 \\ 4 \times 3 + (-1) \times 1 + 1 \times (-5) & 4 \times 2 + (-1) \times 7 + 1 \times 4 \\ 1 \times 3 + 3 \times 1 + (-2) \times (-5) & 1 \times 2 + 3 \times 7 + (-2) \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 40 \\ 6 & 5 \\ 16 & 15 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[5 2 4;4 -1 1;1 3 -2];
>> B=[3 2;1 7;-5 4];
>> A*B

ans =

    -3    40
     6     5
    16    15
```

Example B.7

a) Computer the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 9 & 0 \\ 2 & 8 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

b) Also use a cofactor expansion across the third row to compute the determinant of A.

Solution:

a)
$$\det \mathbf{A} = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$\det \mathbf{A} = 1 \det \begin{bmatrix} 8 & -1 \\ -2 & 0 \end{bmatrix} - 9 \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \det \begin{bmatrix} 2 & 8 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0 - 2) - 9\,(0 - 0) + 0(-4 - 0) = -2$$

b) Using the cofactor expansion method:

$$\det \mathbf{A} = a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33}$$

$$= (-1)^{3+1}a_{31} \det A_{31} + (-1)^{3+2}a_{32} \det A_{32} + (-1)^{3+3}a_{33} \det A_{33}$$

$$= 0 \begin{vmatrix} 9 & 0 \\ 8 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 9 \\ 2 & 8 \end{vmatrix}$$

$$= 0 + 2(-1) + 0 = -2$$

Example B.8

Given

$$\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

Show that these matrices do not commute. That is, verify that $\mathbf{AB} \neq \mathbf{BA}$.

Solution:

$$\mathbf{AB} = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

Clearly, $\mathbf{AB} \neq \mathbf{BA}$.

MATLAB Solution:

```
>> A=[5 1;3 -2];
>> B=[2 0;4 3];
>> A*B

ans =

    14     3
    -2    -6

>> B*A

ans =

    10     2
    29    -2
```

Example B.8

Obtain the determinant of the following square 4 × 4 matrix.

$$\mathbf{A} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

Solution:

The determinant is given by

$$|\mathbf{A}| = \mathrm{a}\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - \mathrm{b}\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + \mathrm{c}\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - \mathrm{d}\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Hence $|\mathbf{A}| = \mathrm{a}[(\mathrm{fkp} + \mathrm{g}\,l\,\mathrm{n} + \mathrm{joh}) - (\mathrm{hkn} + \mathrm{gjp} + l\,\mathrm{of})]$

$- \mathrm{b}[(\mathrm{ekp} + \mathrm{g}\,l\,\mathrm{m} + \mathrm{ioh}) - (\mathrm{hkm} + \mathrm{gip} + l\,\mathrm{oe})]$

$+ \mathrm{c}[(\mathrm{ejp} + \mathrm{f}\,l\,\mathrm{m} + \mathrm{inh}) - (\mathrm{hjm} + \mathrm{fip} + l\,\mathrm{ne})]$

$- \mathrm{d}[(\mathrm{ejo} + \mathrm{fkm} + \mathrm{ing}) - (\mathrm{gjm} + \mathrm{fio} + \mathrm{kne})]$

Example B.9

If $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 5 & 6 \end{bmatrix}$, show that $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$.

Solution:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 2 \\ 5 & 6 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 24 & 26 \\ 23 & 22 \end{bmatrix}$$

$$(\mathbf{AB})^{\mathbf{T}} = \begin{bmatrix} 24 & 23 \\ 26 & 22 \end{bmatrix}$$

$$\mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 23 \\ 26 & 22 \end{bmatrix}$$

Clearly, $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$.

MATLAB Solution:

```
>> A=[1 4;2 3];
>> B=[4 2;5 6];
>> (A*B)'

ans =

    24    23
    26    22

>> B'*A'

ans =

    24    23
    26    22
```

Example B.9

Calculate the determinant of the following 4 × 4 square matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 1 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

Solution:

The determinant is obtained using a first-row expansion.

$$|\mathbf{A}| = (1)\begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & -2 & 1 \end{vmatrix} -5\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 2 & 0 & -2 \end{vmatrix}$$

Expansion by the first row is used to evaluate each of the 3×3 determinants. That is

$$|\mathbf{A}| = (2)\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - 5\left[(1)\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix}\right]$$

$$|\mathbf{A}| = (2)[(3)(1) - (1)(-2)] + [(-1)(1) - (1)(0)]$$
$$- (5)\{[(-1)(-2) - (3)(0)] - (2)[(2)(-2) - (3)(2)]$$
$$- [(2)(0) - (-1)(2)]\}$$

Hence, $|\mathbf{A}| = 9 + 5(-20) = -91$

Example B.10

Find the transpose of matrix A

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 9 & 8 \\ 8 & 7 & 4 & 2 \end{bmatrix}$$

Solution:

The trasnspose of a matrix is constructed by taking the i^{th} row and making it the i^{th} column. Hence,

$$\mathbf{A}^T = \begin{bmatrix} 5 & 8 \\ 8 & 7 \\ 9 & 4 \\ 8 & 2 \end{bmatrix}.$$

MATLAB Solution:

```
>> A=[5 8 9 8;8 7 4 2];
>> A'

ans =

     5     8
     8     7
     9     4
     8     2
```

Example B.10

Consider a third-order determinant, A given by

$$\mathbf{A} = \begin{vmatrix} 2 & 1 & 7 \\ 4 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix}$$

The minor of the term $a_{21} = 4$ is

$$\mathbf{M}_{21} \text{ of } \begin{vmatrix} 2 & 1 & 7 \\ 4 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ 0 & 3 \end{vmatrix} = 3$$

and its cofactor is

$$\mathbf{C}_{21} = (-1)^{2+1}\, 3 = -3$$

The given determinant can be expanded as follows:

$$\mathbf{A} = \begin{vmatrix} 2 & 1 & 7 \\ 4 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(-1)^{1+2} \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + 2(-1)^{2+2} \begin{vmatrix} 2 & 7 \\ 2 & 3 \end{vmatrix} + 0(-1)^{3+2} \begin{vmatrix} 2 & 7 \\ 4 & 1 \end{vmatrix}$$

$$= -10 - 16 = -26$$

Example B.11

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

Show that $\mathbf{A}(\text{adj } \mathbf{A}) = |\mathbf{A}|\, \mathbf{I}$.

Solution:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = 17$$

$$\text{adj } \mathbf{A} = \begin{bmatrix} \begin{vmatrix} -1 & -2 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix}$$

Hence, $\mathbf{A}(\text{adj } \mathbf{A}) = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix}$

$$= 17 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |\mathbf{A}|\, \mathbf{I}$$

MATLAB Solution:

```
>> A=[1 2 0;3 -1 -2;1 0 -3];
>> adjA=inv(A)*det(A)*1

adjA =

  3.0000e+000  6.0000e+000 -4.0000e+000
  7.0000e+000 -3.0000e+000  2.0000e+000
  1.0000e+000  2.0000e+000 -7.0000e+000

>> A*adjA

ans =

  1.7000e+001            0            0
 -1.5543e-015  1.7000e+001  1.7764e-015
            0            0  1.7000e+001

>> det(A)*1

ans =

    17

>> Hence, they are the same.
```

Example B.11

If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, then

$$\text{adj } \mathbf{A} = \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$$

and $\quad \det \mathbf{A} = 2 \times 1 - 5 \times 3 = -13$

$$\text{Hence, } \mathbf{A}^{-1} = -\frac{1}{13}\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} & \frac{3}{13} \\ \frac{5}{13} & -\frac{2}{13} \end{bmatrix}$$

Example B.12

Given two m × n matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 6 & 3 \\ 8 & 2 & 14 \end{bmatrix}$$

Find their sum $\mathbf{A} + \mathbf{B}$.

Solution:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 3 \\ 8 & 2 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 8 & 0 \\ 12 & 2 & 16 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[1 2 -3;4 0 2];
>> B=[-1 6 3;8 2 14];
>> A+B

ans =

     0     8     0
    12     2    16
```

Example B.12

Obtain the solution of the following simultaneous equations by Cramer's rule:

a) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -3 & 2 \\ 3 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$

Solution:

a) $x_i = \dfrac{|B_i|}{|A|}$

$$x_1 = \frac{\begin{vmatrix} 5 & 3 \\ 12 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}} = \frac{-5-36}{-13} = 3.15$$

$$x_2 = \frac{\begin{vmatrix} 1 & 5 \\ 4 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}} = \frac{12-20}{-13} = 0.62$$

b)

$$x_i = \frac{|B_i|}{|A|}$$

$$x_1 = \frac{\begin{vmatrix} 8 & -3 & 2 \\ 5 & 4 & 1 \\ 2 & 2 & -9 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 2 \\ 3 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix}} = \frac{-441}{-63} = 7$$

$$x_2 = \frac{\begin{vmatrix} 1 & 8 & 2 \\ 3 & 5 & 1 \\ -4 & 2 & -9 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 2 \\ 3 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix}} = \frac{189}{-63} = -3$$

$$x_3 = \frac{\begin{vmatrix} 1 & -3 & 8 \\ 3 & 4 & 5 \\ -4 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 2 \\ 3 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix}} = \frac{252}{-63} = -4$$

Example B.13

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 2 \\ 3 & -1 & a \end{bmatrix}$$

Find their sum $\mathbf{A} + \mathbf{B}$.

Solution:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 2 \\ 3 & -1 & a \end{bmatrix}$$

We can find that the sum as

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+1 & 2-3 & 0+2 \\ 3+0 & -1+1 & -2+2 \\ 1+3 & 0-1 & -3+a \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 3 & 0 & 0 \\ 4 & -1 & -3+a \end{bmatrix}$$

MATLAB Solution:

```
>> syms a
>> A=[1 2 0; 3 -1 -2;1 0 -3];
>> B=[1 -3 2;0 1 2;3 -1 a];
>> A+B

ans =

[ 2,   -1,     2]
[ 3,    0,     0]
[ 4,   -1, -3+a]
```

Example B.13

Obtain the solution of the following simultaneous equations by the matrix inverse method.

a) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$

Solution:

a) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

$$C_{11} = (-1)^{1+1}|-1| = -1$$

$$C_{12} = (-1)^{1+2}|4| = -4$$

$$C_{21} = (-1)^{2+1}|3| = -3$$

$$C_{22} = (-1)^{2+2}|1| = 1$$

Hence, $$\mathbf{C} = \begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^{\mathrm{T}}}{|\,\mathbf{A}\,|} = \frac{-1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$$

Hence, $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$= \frac{-1}{13}\begin{bmatrix} -5-36 \\ -20+12 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} -41 \\ -8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{-41}{-13} = 3.15$$

and $$x_2 = \frac{-8}{-13} = 0.62$$

b) $$|A| = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix} = 40$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{bmatrix} 5 & -5 & 10 \\ 10 & -2 & -4 \\ -5 & 13 & 6 \end{bmatrix}$$

The transpose of **C** is the adjoint of **A**

or

$$\text{Adj } \mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 5 & 10 & -5 \\ -5 & -2 & 13 \\ 10 & -4 & 6 \end{bmatrix}$$

Hence, $$\mathbf{A}^{-1} = \text{Adj } \mathbf{A}/|A| = \frac{1}{40}\begin{bmatrix} 5 & 10 & -5 \\ -5 & -2 & 13 \\ 10 & -4 & 6 \end{bmatrix}$$

Therefore,

$$X = \mathbf{A}^{-1}Y = \frac{1}{40}\begin{bmatrix} 5 & 10 & -5 \\ -5 & -2 & 13 \\ 10 & -4 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 0 \\ 40 \\ 80 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

or $x_1 = 0, x_2 = 1, x_3 = 2$

Example B.14

Find the determinant of matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 8 & 16 \\ 2 & 4 \end{bmatrix}$$

Solution:

For square 2 × 2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 8 & 16 \\ 2 & 4 \end{vmatrix} = 8 \times 4 - 16 \times 2 = 0$$

MATLAB Solution:

```
>> A=[8 16;2 4];
>> det(A)

ans =

     0
```

Example B.14

Consider the matrix A given by

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 3 & 2 & 8 \\ 2 & 3 & 9 \end{bmatrix}$$

Because **A** is a 3×3 square matrix, there will be 3 eigenvalues for **A**. The characteristic equation is given by

$$\begin{vmatrix} 7-\lambda & 3 & 4 \\ 3 & 2-\lambda & 8 \\ 2 & 3 & 9-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\,[(2-\lambda)\,(9-\lambda) - 24] - 3\,[3(9-\lambda) - 16] + 4\,[9 - 2(2-\lambda)] = 0$$

Simplifying and solving the above equation, we get three roots for the characteristic equation. These three roots are the eigenvalues of the matrix **A** and is given by

$$\lambda_1 = 5$$
$$\lambda_2 = 13.797$$
$$\lambda_3 = -0.797$$

Substitution λ_1 in Equation (2.59), we get the eigenvectors corresponding to λ_i. Hence,

$$\begin{bmatrix} 7 & 3 & 4 \\ 3 & 2 & 8 \\ 2 & 3 & 9 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \\ V_{13} \end{bmatrix} = 5\begin{bmatrix} V_{11} \\ V_{12} \\ V_{13} \end{bmatrix}$$

From the above equation, we get three linear simultaneous equations as:

$$7V_{11} + 3V_{12} + 4V_{13} = 5V_{11}$$
$$3V_{11} + 2V_{12} + 8V_{13} = 5V_{12}$$
$$2V_{11} + 3V_{12} + 9V_{13} = 5V_{13}$$

Solving these linear simultaneous equations, we get the eigenvector corresponding to $\lambda_1 = 5$ as

$$\mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{12} \\ V_{13} \end{bmatrix} = \begin{bmatrix} -0.9183 \\ 0.1021 \\ 0.3826 \end{bmatrix}$$

Similarly, the eigenvectors corresponding to λ_2 and λ_3 can be calculated as:

$$\mathbf{V}_2 = \begin{bmatrix} V_{21} \\ V_{22} \\ V_{23} \end{bmatrix} = \begin{bmatrix} 0.5903 \\ 0.5504 \\ 0.5903 \end{bmatrix}$$

and

$$\mathbf{V}_3 = \begin{bmatrix} V_{31} \\ V_{32} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 0.2393 \\ -0.9410 \\ 0.2393 \end{bmatrix}$$

Example B.15

The determinant of matrix **A** is −15. Find the missing value of element “a”.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 2 \\ 3 & -1 & a \end{bmatrix}$$

Solution:

Calculate the determinant by expanding the matrix along its first column. The second term drops out because its coefficient is zero.

$$|\mathbf{A}| = (1)(1 \times a + 1 \times 2) + 0 + (3)\ (-3 \times 2 - 2 \times 1) - 15$$
$$= (a + 2) + 3(-8) = a - 22$$

Therefore, a = 7

Example B.15

Consider the matrix A given by:

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 2 & 1 \\ 2 & 3 & 9 \end{bmatrix}$$

The principal minors of **A** are

$$\mathbf{A}_1 = 5 > 0$$

$$\mathbf{A}_2 = \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} = 1 > 0$$

$$\mathbf{A}_3 = \begin{vmatrix} 5 & 3 & 4 \\ 3 & 2 & 1 \\ 2 & 3 & 9 \end{vmatrix} = 20 > 0$$

Because all the principal minors of **A** are positive, the matrix **A** is positive definite matrix.

Example B.16

If the determinant of matrix **A** is –40, what is the determinant of **B**?

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 6 & 4 & 2 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution:

The first row of matrix **B** is twice that of matrix **A**. The determinant of **B** is twice the determinant of **A**.

Example B.16

Consider the matrix A given by

$$\mathbf{A} = \begin{vmatrix} -8 & 4 \\ 1 & -5 \end{vmatrix}$$

The principal minors of **A** are

$$\mathbf{A}_1 = -8 < 0$$

$$\mathbf{A}_2 = \begin{vmatrix} -8 & 4 \\ 1 & -5 \end{vmatrix} = 36 > 0$$

Because the principal minors A_1 is negative and A_2 is positive, the matrix **A** is negative definite matrix.

Example B.17

Find the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution:

The determinant may be computed using the third row because it contains a zero and hence, fewer calculations will be involved. Therefore,

$$|\mathbf{A}| = 2(-1)^{3+1}M_{31} + 0M_{32} + 1(-1)^{3+3}M_{33}$$

$$= 2\begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} = 2(2-3) + (-1-8) = -11$$

indicating that **A** is nonsingular. The same result can also be obtained using the second column, which also contains a zero.

MATLAB Solution:

```
>> A=[1 2 -3;4 -1 1;2 0 1];
>> det(A)

ans =

   -11
```

Example B.17

Consider the matrix **A** given by:

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix}$$

The principal minors of **A** are

$$A_1 = 3 > 0$$

$$A_2 = \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix} = -22 < 0$$

Because the principal minors A_1 is positive and A_2 is negative, the matrix **A** is indefinite matrix.

Example B.18

Evaluate

$$\begin{vmatrix} 3 & -1 & -2 \\ 5 & 0 & -4 \\ 2 & -3 & 6 \end{vmatrix}$$

Solution:

Expand by minors about the first column

$$\begin{vmatrix} 3 & -1 & -2 \\ 5 & 0 & -4 \\ 2 & -3 & 6 \end{vmatrix} = 3\begin{vmatrix} 0 & -4 \\ -3 & 6 \end{vmatrix} - 5\begin{vmatrix} -1 & 2 \\ -3 & 6 \end{vmatrix} + 2\begin{vmatrix} -1 & -2 \\ 0 & -4 \end{vmatrix}$$

$$= 3[0(6) - (-3)(-4)] - 5[(-1)6 - (-3)(-2)] + 2[(-1)(-4) - 0(-2)]$$

$$= 3[-12] - 5[-12] + 2[4] = -36 + 60 + 8 = 32$$

MATLAB Solution:

```
>> A=[3 -1 -2;5 0 -4;2 -3 6];
>> det(A)

ans =

    32
```

Example B.19

Determine the adjoint of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 0 & -1 \\ -2 & 3 & -4 \end{bmatrix}$$

Solution:

$$[\text{adj } \mathbf{A}] = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 3 & -4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ -2 & -4 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ -2 & 3 \end{vmatrix} \\ \begin{vmatrix} 4 & -2 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 14 & 9 \\ 22 & 0 & -11 \\ -4 & 7 & -12 \end{bmatrix}^T = \begin{bmatrix} 3 & 22 & -4 \\ 14 & 0 & 7 \\ 9 & -11 & -12 \end{bmatrix}$$

MATLAB Solution:

```
>> I=[1 0 0;0 1 0;0 0 1];
>> A=[1 4 2;3 0 -1;-2 3 -4];
>> adjA=inv(A)*det(A)*I

adjA =

  3.0000e+000  2.2000e+001 -4.0000e+000
  1.4000e+001 -2.5906e-016  7.0000e+000
  9.0000e+000 -1.1000e+001 -1.2000e+001
```

Example B.20

Determine the inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Solution:

First, calculate the determinant using the first row as follows:

$$|\mathbf{A}| = -\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 10$$

Next, identify the minor corresponding to each entry to form the adjoint matrix; we have,

$$\text{adj}\,(\mathbf{A}) = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

Hence, $\mathrm{A}^{-1} = \dfrac{1}{10}\,\text{adj}\,(\mathbf{A}) = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$

MATLAB Solution:

```
>> format short
>> A=[-1 1 2;3 -1 1;-1 3 4];
>> inv(A)

ans =

   -0.7000    0.2000    0.3000
   -1.3000   -0.2000    0.7000
    0.8000    0.2000   -0.2000
```

Example B.21

The cofactor matrix of matrix **A** is **C**. Find the inverse of matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 6 & -8 & -1 \\ -5 & 10 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

Solution:

The classical adjoint is the transpose of the cofactor matrix.

$$\text{adj}\,(\mathbf{A}) = \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 6 & -5 & -2 \\ -8 & 10 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Calculate the determinant of **A** by expanding along the top row

$$|\mathbf{A}| = (4)(8-2) - (2)(12-4) + (3)(3-4) = 5.$$

Divide the classical adjoint by the determinant

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{\begin{bmatrix} 6 & -5 & -2 \\ -8 & 10 & 1 \\ -1 & 0 & 2 \end{bmatrix}}{5}$$

Hence, $$\mathbf{A}^{-1} = \begin{bmatrix} 1.2 & -1.0 & -0.40 \\ -1.6 & 2.0 & 0.20 \\ -0.20 & 0 & 0.40 \end{bmatrix}$$

Example B.22

Given that matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$, find the inverse of **A**.

Solution:

Referring to Example B.11, we have found that

$$\operatorname{adj}\,\mathbf{A} = \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix} \quad \text{and} \quad |\mathbf{A}| = 17$$

Hence, the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{|\,\mathbf{A}\,|} = \frac{1}{17}\begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[1 2 0;3 -1 -2;1 0 -3];
>> inv(A)

ans =

    0.1765    0.3529   -0.2353
    0.4118   -0.1765    0.1176
    0.0588    0.1176   -0.4118
```

Example B.23

Given $\mathbf{A} = \begin{bmatrix} 10 & 3 & 10 \\ 8 & -2 & 9 \\ 8 & 1 & -10 \end{bmatrix}$, determine $\mathbf{A}^{-1}$.

MATLAB Solution:

```
>> A=[10 3 10;8 -2 9;8 1 -10];
>> inv(A)

ans =

    0.0136    0.0496    0.0583
    0.1886   -0.2233   -0.0124
    0.0298    0.0174   -0.0546
```

Example B.24

Determine the inverse of the given square 3 × 3 matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[1 2 3;4 5 6; 7 0 0];
>> inv(A)

ans =

         0         0    0.1429
   -2.0000    1.0000   -0.2857
    1.6667   -0.6667    0.1429
```

Example B.25

Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution:

a) The determinant of **A** is $|\mathbf{A}| = 3$

b) The minors of **A** are given by

$$M_{11} = \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6, \quad M_{12} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \ldots$$

c) Apply the signs $(-1)^{i+j}$ to the minors to form the cofactors

$$[\mathbf{C}_{ij}] = \begin{vmatrix} 6 & -1 & -2 \\ -3 & 2 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

d) The adjoint matrix is the transpose of the cofactor matrix, or $[\mathbf{C}_{ij}]^T = [\mathbf{C}_{ji}]$. Thus, the inverse $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \frac{1}{|\,\mathbf{A}\,|}\,\text{adj}\,\mathbf{A} = \frac{1}{3}\begin{bmatrix} 6 & -3 & 0 \\ -1 & 2 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

e) The result can be verified as follows:

$$\mathbf{A}^{-1}\mathbf{A} = \frac{1}{3}\begin{bmatrix} 6 & -3 & 0 \\ -1 & 2 & -1 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It should be noted that for an inverse to exist, the determinant $|\mathbf{A}|$ must not be zero.

MATLAB Solution:

```
>> A=[1 1 1;1 2 2;1 0 3];
>> inv(A)

ans =

    2.0000   -1.0000        0
   -0.3333    0.6667   -0.3333
   -0.6667    0.3333    0.3333
```

Example B.26

Solve the following set of simultaneous linear equations for x, y and z.

$$2x + 3y - z = -10$$
$$-x + 4y + 2z = -4$$
$$2x - 2y + 5z = 35$$

Solution:

Using Cramer's rule to solve the simultaneous linear equations, the coefficient matrix is

$$\mathbf{D} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 2 & -2 & 5 \end{bmatrix}$$

The determinant is

$$|\mathbf{D}| = 2\begin{vmatrix} 4 & 2 \\ -2 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= 2(20 + 4) + (1)(15 - 2) + (2)(6 + 4) = 81$$

The determinant of the substitutional matrices is

$$|\mathbf{A_1}| = \begin{vmatrix} -10 & 3 & -1 \\ -4 & 4 & 2 \\ 35 & -2 & 5 \end{vmatrix} = 162$$

$$|\mathbf{A_2}| = \begin{vmatrix} 2 & -10 & -1 \\ -1 & -4 & 2 \\ 2 & 35 & 5 \end{vmatrix} = -243$$

$$|\mathbf{A_3}| = \begin{vmatrix} 2 & 3 & -10 \\ -1 & 4 & -4 \\ 2 & -2 & 35 \end{vmatrix} = 405$$

$$x = \frac{162}{81} = 2$$

$$y = \frac{-243}{81} = -3$$

$$z = \frac{405}{81} = 5$$

MATLAB Solution:

```
>> A=[2 3 -1;-1 4 2;2 -2 5];
>> b=[-10;-4;35];
>> x=A\b

x =

    2.0000
   -3.0000
    5.0000

>> x=inv(A)*b

x =

    2.0000
   -3.0000
    5.0000
```

Example B.27

Solve the following system of linear equations for x, y, and z.

$$10x + 3y + 10z = 5$$

$$8x - 2y + 9z = 2$$

$$8x + y - 10z = 7$$

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Solution:

There are several ways to solve this problem.

$$\mathbf{A\ X = B}$$

$$\begin{bmatrix} 10 & 3 & 10 \\ 8 & -2 & 9 \\ 8 & 2 & -10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

$$\mathbf{A\ A^{-1}\ X\ = A^{-1}\ B}$$

$$\mathbf{I\ X = A^{-1}B}$$

or $\mathbf{X = A^{-1}B}$

$$\mathbf{X} = \begin{bmatrix} \frac{1}{398} & \frac{25}{398} & \frac{47}{796} \\ \frac{38}{398} & \frac{-45}{199} & \frac{-5}{398} \\ \frac{8}{199} & \frac{1}{199} & \frac{-11}{199} \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} (5)\left(\frac{1}{398}\right) + (2)\left(\frac{25}{398}\right) + (7)\left(\frac{47}{796}\right) \\ (5)\left(\frac{38}{398}\right) + (2)\left(\frac{-45}{199}\right) + (7)\left(\frac{-5}{398}\right) \\ (5)\left(\frac{8}{199}\right) + (2)\left(\frac{1}{199}\right) + (7)\left(\frac{-11}{199}\right) \end{bmatrix}$$

or $$\mathbf{X} = \begin{bmatrix} 0.01256 + 0.12562 + 0.41331 \\ 0.47738 - 0.45226 - 0.08793 \\ 0.20100 + 0.01005 - 0.38693 \end{bmatrix} = \begin{bmatrix} 0.55149 \\ -0.06281 \\ -0.17588 \end{bmatrix}$$

Hence, $x = 0.55149$, $y = -0.06281$ and $z = -0.17588$ is the solution.

MATLAB Solution:

```
>> A=[10 3 10;8 -2 9;8 2 -10];
>> B=[5;2;7];
>> x=A\B

x =

    0.5515
    0.4146
   -0.1759

>> x=inv(A)*B

x =

    0.5515
    0.4146
   -0.1759
```

Example B.28

For the following system of equations, determine the classical adjoint of the coefficient matrix.

$$10x + 3y + 10z = 5$$
$$8x - 2y + 9z = 5$$
$$8x + 2y - 10z = 5$$

Solution:

The entries in the cofactor matrix are the determinants of submatrices resulting from elimination of row i and column j for entry a_{ij}. The entry is multiplied by +1 or −1 depending on its position. The coefficient matrix is

$$\begin{bmatrix} 10 & 3 & 10 \\ 8 & -2 & 9 \\ 8 & 2 & -10 \end{bmatrix}$$

The entries in the cofactor matrix are

$$a_{11} = (+1)[(-2)(-10) - (9)(2)] = 2$$
$$a_{12} = (-1)[(8)(-10) - (9)(8)] = 152$$
$$a_{13} = (+1)[(8)(2) - (-2)(8)] = 32$$
$$a_{21} = (-1)[(3)(-10) - (10)(2)] = 50$$
$$a_{22} = (+1)[(10)(-10) - (10)(8)] = 4$$
$$a_{23} = (-1)[(10)(2) - (3)(8)] = 4$$
$$a_{31} = (+1)[(3)(9) - (10)(-2)] = 47$$
$$a_{32} = (-1)[(10)(9) - (10)(8)] = -10$$
$$a_{33} = (+1)[(10)(-2) - (3)(8)] = -44$$

The cofactor matrix is

$$\begin{bmatrix} 2 & 152 & 32 \\ 50 & -180 & 4 \\ 47 & -10 & -44 \end{bmatrix}$$

The classical adjoint is the transpose of the cofactor matrix.

$$\mathbf{A}_{adj} = \begin{bmatrix} 2 & 50 & 47 \\ 152 & -180 & -10 \\ 32 & 4 & -44 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[10 3 10;8 -2 9;8 2 -10];
>> I=[1 0 0;0 1 0;0 0 1];
>> adjA=inv(A)*det(A)*I

adjA =

    2.0000   50.0000   47.0000
  152.0000 -180.0000  -10.0000
   32.0000    4.0000  -44.0000
```

Example B.29

Solve the following system of three equations in three unknowns:

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 1 \\ -x_1 + 2x_2 + x_3 = 8 \\ x_1 - 3x_2 - 2x_3 = -13 \end{cases}$$

Solution:

Following the general formulation and the information on the determinant of a matrix we proceed as

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & 1 \\ 1 & -3 & -2 \end{vmatrix} = -6, \qquad \Delta_1 = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 2 & 1 \\ -13 & -3 & -2 \end{vmatrix} = 6$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 8 & 1 \\ 1 & -13 & -2 \end{vmatrix} = -12, \qquad \Delta_3 = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 8 \\ 1 & -3 & -13 \end{vmatrix} = -18$$

Subsequently, the three unknown quantities are determined as

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{6}{-6} = -1$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{-12}{-6} = 2$$

$$x_3 = \frac{\Delta_1}{\Delta} = \frac{-18}{-6} = 3$$

MATLAB Solution:

```
>> A=[2 3 -1;-1 2 1;1 -3 -2];
>> b=[1;8;-13];
>> x=A\b

x =

   -1.0000
    2.0000
    3.0000
```

```
>> x=inv(A)*b

x =

   -1.0000
    2.0000
    3.0000
```

Example B.30

Solve the system of equations using the matrix inversion method.

$$3x_1 + 4x_2 = 3$$

$$5x + 6x_2 = 7$$

Solution:

The system is equivalent to **Ax** = **b**. Hence,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

where $\det(A) = 3(6) - 4(5) = -2 \neq 0.$

Hence, A is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[3 4;5 6];
>> b=[3;7];
>> x=A\b

x =

    5.0000
   -3.0000

>> x=inv(A)*b

x =

    5.0000
   -3.0000
```

Example B.31

Solve the following system of equations:

$$2x_1 - x_2 + 3x_3 = 4$$
$$x_1 + 9x_2 - 2x_3 = -8$$
$$4x_1 - 8x_2 + 11x_3 = 15$$

Solution:

The matrix of coefficients is

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 9 & -2 \\ 4 & -8 & -11 \end{bmatrix}$$

The inverse of A is

$$\mathbf{A}^{-1} = \frac{1}{53}\begin{bmatrix} 83 & -13 & -25 \\ -19 & 10 & 7 \\ -44 & 12 & 19 \end{bmatrix}$$

Hence, the solution of the system is

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{53}\begin{bmatrix} 83 & -13 & -25 \\ -19 & 10 & 7 \\ -44 & 12 & 19 \end{bmatrix}\begin{bmatrix} 4 \\ -8 \\ 15 \end{bmatrix} = \begin{bmatrix} 61/53 \\ -51/53 \\ 13/53 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[2 -1 3;1 9 -2;4 -8 -11];
>> b=[4;-8;15];
>> x=A\b

x =

    1.5205
   -1.0658
   -0.0356

>> x=inv(A)*b

x =

    1.5205
   -1.0658
   -0.0356
```

Example B.32

Determine the unique solution of the following system of equations:

$$x_1 - x_2 + 3x_3 - x_4 = 1$$
$$0 + x_2 - 3x_3 + 5x_4 = 2$$
$$x_1 + 0 - x_3 + x_4 = 0$$
$$x_1 + 2x_2 + 0 - x_4 = -5$$

Solution:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{33}\begin{bmatrix} 5 & -3 & 24 & 4 \\ 1 & 6 & -15 & 14 \\ 12 & 6 & -15 & 3 \\ 7 & 9 & -6 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -7 \\ -19 \\ 3 \\ 10 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[1 -1 3 -1;0 1 -3 5;1 0 -1 1;1 2 0 -1];
>> b=[1;2;0;-5];
>> x=A\b

x =

   -0.6364
   -1.7273
    0.2727
    0.9091

>> x=inv(A)*b

x =

   -0.6364
   -1.7273
    0.2727
    0.9091
```

Example B.33

Determine the eigenvalues and the corresponding eigenvectors of the following 2 × 2 matrix.

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

Solution:

The characteristic equation is given by

$$|\mathbf{A} - \lambda\,\mathbf{I}| = 0$$

or
$$\begin{vmatrix} -1-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} = 0 \tag{1}$$

which gives

$$(\lambda + 1)\,(\lambda - 2) = 0$$

or
$$\lambda_{1,2} = -1,\, 2 \tag{2}$$

The eigenvectors are given by solving the equation

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{V}_1 = 0$$

or
$$(\mathbf{A} + \mathbf{I})\mathbf{V_1} = 0 \tag{3}$$

where $\mathbf{V_1}$ is the 2 × 1 eigenvector corresponding to $\lambda_1 = -1$. That is,

$$\mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \tag{4}$$

Rewriting Eq. (3) as

$$\begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

Eq. (5) shows that the two rows of the matrix A + I are linearly dependent and that there is a free variable. Hence,

$$-3\, V_{21} = 0$$

or
$$V_{21} = 0 \tag{6}$$

If V_{11} is the free variable, then the eigenvector V_1 is given by

$$\mathbf{V}_1 = \begin{bmatrix} V_{11} \\ 0 \end{bmatrix} \tag{7}$$

Because V_{11} is arbitrary, we can assign a value of 1. Hence,

$$\mathbf{V}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8}$$

In a similar manner, the eigenvector corresponding to $\lambda_1 = 2$ can be written as

$$\mathbf{V}_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \tag{9}$$

and
$$(\mathbf{A} + \lambda_2\mathbf{I})\mathbf{V_2} = 0$$

or
$$(\mathbf{A} + 2\mathbf{I})\mathbf{V_2} = 0 \tag{10}$$

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

or
$$V_{12} = -V_{22} \tag{12}$$

If V_{22} is the free variable, (equal to 1), then the eigenvector corresponding to λ_2 is

$$V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{13}$$

Hence, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \tag{14}$$

of these vectors can be referred to as the basis of all eigenvectors of matrix **A**.

MATLAB Solution:

```
>> A=[-1 -3;0 2];
>> eig(A)

ans =

    -1
     2

>> [Q,D]=eig(A)

Q =

    1.0000    -0.7071
    0          0.7071


D =

    -1      0
     0      2
```

Example B.34

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Solution:

The characteristic equation is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & 1 & 2 \\ 1 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (4-\lambda)\lambda^2 + \lambda + 4\lambda = 0 \qquad (1)$$

Eq. (1) can be rewritten as

$$\lambda(\lambda - 5)(\lambda + 1) = 0 \qquad (2)$$

The roots of Eq. (2) are

$$\lambda_1 = 0, \quad \lambda_2 = 5, \quad \lambda_3 = -1 \qquad (3)$$

The ith eigenvector associate with the eigenvalue λi can be obtained using the following equation.

$$(\mathbf{A} + \lambda_i\ \mathbf{I})\mathbf{y}_i = 0 \qquad (4)$$

The solution of Eq. (4) gives

$$\mathbf{y}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \qquad (5)$$

MATLAB Solution:

```
>> A=[4 1 2;1 0 0;2 0 0];
>> eig(A)

ans =

   -1.0000
   -0.0000
    5.0000

>> [Q,D]=eig(A)

Q =

    0.4082    0.0000    0.9129
   -0.4082   -0.8944    0.1826
   -0.8165    0.4472    0.3651


D =

   -1.0000         0         0
         0   -0.0000         0
         0         0    5.0000
```

Example B.35

Determine the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

Solution:

The eigenvalues of **A** are determined by finding the values of λ satisfying the condition

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0 \tag{1}$$

Expansion of Eq. (1) gives

$$(2-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -2 \\ 0 & 3-\lambda \end{vmatrix} = 0 \tag{2}$$

When the 2×2 determinants in Eq. (2) are expanded, the following cubic equation is obtained.

$$-\lambda^3 + 8\lambda^2 - 16\lambda + 7 = 0 \tag{3}$$

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The eigenvalues are the roots of the cubic equation that are found to be 0.609, 2.227, and 5.164. The eigenvector corresponding to the smallest eigenvalue is given by

$$\begin{bmatrix} 1.391 & -1 & 0 \\ -1 & 2.391 & -2 \\ 0 & -2 & 2.391 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

The first equation gives $x_1 = 0.719x_2$. The third equations gives $x_3 = 0.836x_2$. Substituting these values into the second equation, it is identically satisfied. Hence, x_2 remains arbitrary and the eigenvector of **A** corresponding to $\lambda = 0.609$ is

$$C_1 \begin{bmatrix} 0.719 \\ 1 \\ 0.836 \end{bmatrix} \tag{5}$$

where C_1 is an arbitrary constant. The same procedure is followed giving the eigenvectors corresponding to the second and third eigenvalues. They are

$$C_2 \begin{bmatrix} -4.41 \\ 1 \\ 2.59 \end{bmatrix} \qquad C_3 \begin{bmatrix} -0.316 \\ 1 \\ -0.924 \end{bmatrix} \tag{6}$$

respectively.

It should be noted here that if **A** is an $n \times n$ singular matrix, then one of its eigenvalues is zero. If **A** is nonsingular, then the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of **A**. The eigenvectors of $\mathbf{A}^{-1}$ are the same as the eigenvectors of **A**.

MATLAB Solution:

```
>> A=[2 -1 0;-1 3 -2;0 -2 3];
>> eig(A)

ans =

    0.6086
    2.2271
    5.1642

>> [Q,D]=eig(A)

Q =

   -0.4828    0.8460    0.2261
   -0.6718   -0.1922   -0.7154
   -0.5618   -0.4973    0.6611


D =

    0.6086         0         0
    0           2.2271      0
    0              0         5.1642
```

Example B.36

Find the eigenvalues of $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution:

We need to find all scalars λ such that the matrix equation

$$(A - \lambda I)\mathbf{x} = 0$$

has a nontrivial solution. This problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, where

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 6-\lambda \end{bmatrix}$$

The eigenvalues of A are the solution of the equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & 6-\lambda \end{bmatrix} = 0$$

Now $\quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

Hence, $\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - (3)(3)$

$$= -12 + 6\lambda - 2\lambda + \lambda^2 + 9 = \lambda^2 + 4\lambda - 21$$

Setting $\lambda^2 + 4\lambda - 21 = 0$,

we get $(\lambda - 3)(\lambda + 7) = 0$

Hence, the eigenvalues of A are 3 and −7.

MATLAB Solution:

```
>> A=[2 3;3 -6];
>> eig(A)

ans =

    -7
     3
```

Example B.37

Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \tag{1}$$

$$\lambda I - \mathbf{A} = \begin{bmatrix} \lambda - 1 & 2 \\ -2 & \lambda \end{bmatrix} \tag{2}$$

The characteristic polynomial given by

$$\begin{vmatrix} \lambda - 1 & 2 \\ -2 & \lambda \end{vmatrix} = 0 \tag{3}$$

or
$$= (\lambda - 1)\,\lambda + 4 = \lambda^2 - \lambda + 4 = 0 \tag{4}$$

The roots are $(1 \pm i\sqrt{15})/2$. These are the eigenvalues of **A**.

To obtain the eigenvectors associated with $(1 + i\sqrt{15})/2$, we solve the system

$$[\{(1 + i\sqrt{15})/2\}I - A]\,X = 0$$

which is

$$\begin{bmatrix} \dfrac{1 + i\sqrt{15}}{2} - 1 & 2 \\ -2 & \dfrac{1 + i\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

or
$$\frac{-1 + i\sqrt{15}}{2}\,x_1 + 2x_2 = 0 \tag{6}$$

$$-2x_1 + \frac{1 + i\sqrt{15}}{2}\,x_2 = 0 \tag{7}$$

From Eq. (6)

$$x_2 = \frac{1 - i\sqrt{15}}{4}\,x_1 \tag{8}$$

Substituting, this satisfying the Eq. (8) for any x_1. Therefore, setting $x_1 = \alpha$, the general solution of the system is

$$\alpha \begin{bmatrix} 1 \\ \dfrac{1 - \sqrt{15}i}{4} \end{bmatrix} \tag{9}$$

Any such matrix with $\alpha \neq 0$ is an eigenvector corresponding to the eigenvalue $(1 + i\sqrt{15})/2$.

The eigenvectors corresponding to $(1 - i\sqrt{15})/2$ are nontrivial solution of

$$\begin{bmatrix} \dfrac{1 - i\sqrt{15}}{2} & 2 \\ -2 & \dfrac{1 - i\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

Hence, the eigenvectors are

$$\beta \begin{bmatrix} 1 \\ \dfrac{1 + i\sqrt{15}}{4} \end{bmatrix} \tag{11}$$

for any nonzero numbers β.

MATLAB Solution:

```
>> A=[1 -2;2 0];
>> eig(A)

ans =

   0.5000 + 1.9365i
   0.5000 - 1.9365i

>> [Q,D]=eig(A)

Q =

   0.1768 + 0.6847i   0.1768 - 0.6847i
   0.7071             0.7071


D =

   0.5000 + 1.9365i   0
   0                  0.5000 - 1.9365i
```

Example B.38

Determine the eigenvalues and the corresponding eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 16 & 9 \\ -30 & -17 \end{bmatrix}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 16 & 9 \\ -30 & -17 \end{bmatrix}$$

The characteristic equation and the characteristic values are

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 16-\lambda & 9 \\ -30 & -17-\lambda \end{vmatrix} = 0 \quad \text{gives} \quad \lambda_1 = 1, \lambda_2 = -2$$

Corresponding to $\lambda_1 = 1$, we have

$$[\mathbf{A} - \lambda\mathbf{I}]\mathrm{v} = 0 \quad \text{gives} \quad \begin{bmatrix} 15 & 9 \\ -30 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{gives } 5x = -3y$$

showing that there is one free variable, say, y. Letting $y = 5$ gives $x = -3$, and the eigen-vector is given by

$$\mathrm{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Similarly, for $\lambda_2 = -2$, we get

$$[\mathbf{A} - \lambda\mathbf{I}]\mathrm{v} = 0 \text{ gives } \begin{bmatrix} 18 & 9 \\ -30 & -15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives } 2x = -y$$

Letting $y = 2$, the eigenvector is given by

$$\mathrm{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[16 9;-30 -17];
>> eig(A)

ans =

     1
    -2

>> [Q,D]=eig(A)

Q =

    0.5145   -0.4472
   -0.8575    0.8944


D =

     1     0
     0    -2
```

Example B.39

Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

$$\lambda \mathbf{I}_3 - \mathbf{A} = \begin{bmatrix} \lambda - 1 & 1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \tag{1}$$

The characteristic equation is

$$\begin{vmatrix} \lambda - 1 & 1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = 0 \tag{2}$$

or

$$(\lambda - 1)\begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda + 1 \end{vmatrix} = (\lambda - 1)^2(\lambda + 1) \tag{3}$$

The roots are 1, 1, and −1.

To find the eigenvectors associated with 1, solve the system $(I_3 - A)X = 0$, or

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

The general solution of Eq. (4) is

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

and the nontrivial solution ($\alpha \neq 0$) are the eigenvectors associated with eigenvalue 1.

For eigenvectors associated with −1, we solve the system

$$(-\mathbf{I}_3 - \mathbf{A})X = 0$$

or

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

The general solution of this system is

$$\begin{bmatrix} \beta \\ 2\beta \\ -4\beta \end{bmatrix} \tag{6}$$

and these matrices are eigenvectors corresponding to eigenvalue −1 for any nonzero number β.

MATLAB Solution:

```
>> A=[1 -1 0;0 1 1;0 0 -1];
>> eig(A)

ans =

     1
     1
    -1

>> [Q,d]=eig(A)

Q =

    1.0000     1.0000    -0.2182
    0          0.0000    -0.4364
    0          0          0.8729


d =

     1      0      0
     0      1      0
     0      0     -1
```

Example B.40

Determine the eigenvalues and eigenvectors of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution:

The characteristic equation and the eigenvalues are given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 4 & -\lambda & 0 \\ 1 & 2 & -3-\lambda \end{vmatrix} = -\lambda(\lambda + 3)(\lambda - 2)$$

$$= 0 \text{ gives } \lambda_{1,2,3} = 0, 2, -3$$

For $\lambda_1 = 0$, solve $\mathbf{Av} = 0$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first row reads x = 0. Letting z = 2 be the free variable, then we obtain y = 3. The eigenvectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Similarly, for the remaining two eigenvalue, we can write

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = 0 \text{ gives } \begin{bmatrix} 0 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which shows that x = 0 = y and z is arbitrary, say z = 1.

Hence,

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[2 0 0;4 0 0;1 2 3];
>> eig(A)

ans =

     3
     0
     2

>> [Q,d]=eig(A)

Q =

    0          0          0.1826
    0          0.8321     0.3651
    1.0000    -0.5547    -0.9129
```

```
d =

     3     0     0
     0     0     0
     0     0     2
```

Example B.41

Find the eigenvalues and the corresponding eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic equation and the characteristic values are

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 1) = 0 \quad \text{gives} \quad \lambda_1 = \lambda_2 = 1, \lambda_3 = -1.$$

For $\lambda_1 = \lambda_2 = 1$, we have

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \text{ gives} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

showing that there are two free variables. The first row reads $x = y$. Let $y = 0$ and $z = 1$ to get $x = 0$ and then let $y = 1$ and $z = 0$ to get $x = 1$. Hence, the two independent eigenvectors are given by

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, for $\lambda_3 = -1$, we have

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \text{ gives} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{gives } z = 0,\ x = -y$$

Letting y = 1, the third independent eigenvector is given by

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

MATLAB Solution:

```
>> A=[0 1 0;1 0 0;0 0 1];
>> eig(A)

ans =

    -1
     1
     1

>> [Q,d]=eig(A)

Q =

   -0.7071    0.7071         0
    0.7071    0.7071         0
    0         0              1.0000

d =

    -1      0      0
     0      1      0
     0      0      1
```

Appendix C

Laplace Transforms

The Laplace transformation is an operational method for solving linear, time invariant differential equations and corresponding initial and boundary value problems. The process of solution consists of mainly three steps:

1. The given problem is transformed into a simple equation, often called subsidiary equation.
2. The subsidiary equation is solved by purely algebraic manipulations.
3. The solution of the subsidiary equation is transformed back to obtain the solution of the given problem.

In this way, the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. In comparison with the classical method of solving linear differential equations, the Laplace transform method has the advantage that the initial conditions are automatically taken care of, and the homogeneous equation as well as the particular integral is solved in one operation.

Example Problems and Solutions

Example C.1

Determine the Laplace transform of the following function.

$$f(t) = 5e^{-t} + \sin 3t$$

Solution:

$$\begin{aligned} L[5e^{-t} + \sin 3t] &= 5L[e^{-t}] + L[\sin 3t] \\ &= 5\left(\frac{1}{s+1}\right) + \left(\frac{3}{s^2+9}\right) \end{aligned}$$

MATLAB Solution:

```
>> syms t s
>>
>>f=5*exp(-t) + sin(3*t);
>>F=laplace(f,t,s)

F =

5/(1+s)+3/(s^2+9)
```

Example C.2

Find the Laplace transform of f(t) = 10 t for all $t \geq 0$.

Solution:

The Laplace transform of f(t) is written as

$$F(s) = \int_0^\infty 10t\, e^{-st} dt = \frac{10}{s^2}$$

MATLAB Solution:

```
>> syms  t
>> f=10*t
f =
10*t
>> laplace(f)
ans =
10/s^2
```

Example C.3

Find the Laplace transform of f(t) defined by

$$\begin{aligned} f(t) &= 0 \qquad && t < 0 \\ &= te^{-5t} && t \geq 0 \end{aligned}$$

Solution:

Since $$L(t) = G(s) = 1/s^2$$

and $$L[e^{-\alpha t} f(t)] \int_0^\infty e^{-\alpha t}\, f(t)\, e^{-st} dt = F(s+\alpha)$$

$$F(s) = L[te^{-5t}] = G(s+5) = \frac{1}{(s+5)^2}$$

MATLAB Solution:

```
>> syms t s
f=t*exp(-5*t);
F=laplace(f,t,s)
F =

1/(s+5)^2
```

Example C.4

Find the Laplace transform of $f(t) = e^{-at}$ for all $t \geq 0$, where a is a constant.

Solution:

The Laplace transform of f(t) is

$$F(s) = \int_0^{\infty} e^{-at} e^{-st}\, dt$$

$$= \left. \frac{-e^{-(s+a)t}}{s+a} \right|_0^{\infty} = \frac{1}{s+a}$$

MATLAB Solution:

```
>> syms a
>> f=exp(-a*t)
f =
exp(-a*t)
>> laplace(f)
ans =
1/(s+a)
```

Example C.5

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= \sin(\omega t - \theta) && t \geq 0 \end{aligned}$$

where θ is a constant.

Solution:

We know that $\sin(\omega t - \theta) = \sin \omega t \cos \theta - \cos \omega t \sin \theta$

Hence, $L[\sin(\omega t - \theta)] = \cos \theta\, L[\sin \omega t] - \sin \theta\, L[\cos \omega t]$

$$= \cos\theta \frac{\omega}{s^2 + \omega^2} - \sin\theta \frac{s}{s^2 + \omega^2}$$

$$= \frac{\omega \cos\theta - s \sin\theta}{s^2 + \omega^2}$$

Example C.6

Determine the Laplace transform of the ramp function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= At && \text{for } t \geq 0 \end{aligned}$$

where A is a constant.

Solution:

The Laplace transform is

$$L(At) = A\int_0^\infty t\, e^{-st}\, dt$$

Note that

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b u\, du$$

Here $u = t$ and $dv = e^{-st}\, dt$

Therefore,

$$L[At] = A\int_0^\infty t\, e^{-st}\, dt = A\left(t\, \frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt \right)$$

$$= \frac{A}{s}\int_0^\infty t\, e^{-st}\, dt = \frac{A}{s^2}$$

MATLAB Solution:

```
>> syms A
>> f=heaviside(t)*A*t
f =
heaviside(t)*A*t
>> pretty(laplace(f))

                    A
                   ---
                    2
                    s
```

Example C.7

Determine the Laplace transform of $g(t) = -\cos^2 t$

Solution:

$$\dot{g}\,(t) = +2 \sin t \cos t = \sin 2t \tag{1}$$

Because

$$L\,[\,\dot{g}\,(t)] = sF(s) - f(0) = sF(s) - 1 \tag{2}$$

Also

$$L\,[\,\dot{g}\,(t)] = L(\sin 2\, t) = \frac{2}{s^2 + 4} \tag{3}$$

Equating the right hand side of Eqs. (2) and (3), we have

$$sF(s) - 1 = \frac{2}{s^2 + 4}$$

or
$$F(s) = \frac{1}{s}\left[1 + \frac{2}{s^2+4}\right] = \frac{1}{s}\left[\frac{s^2+4+2}{s^2+4}\right] = \frac{s^2+6}{s(s^2+4)}$$

MATLAB Solution:

```
>> syms t
>> f=-cos(t)^2;
>> g=laplace(f);
>> pretty(simplify(g))
                                    2
                               2 + s
                            - ----------
                                  2
                              s (s  + 4)
```

Example C.8

Find the Laplace transform of $f(t) = e^{\alpha t}$.

Solution:

$$L[e^{\alpha t}] = \int_0^\infty e^{\alpha t} e^{-st} dt$$
$$= \frac{1}{(\alpha - s)} e^{(\alpha - s)t}\Big|_0^\infty = \frac{1}{s-\alpha} \qquad s > \alpha$$

MATLAB Solution:

```
>> syms t s a
f=exp(a*t);
F=laplace(f,t,s)

F =

1/(s-a)
```

Example C.9

Find the inverse Laplace transform of

$$F(s) = \frac{3}{(s+1)(s+2)}$$

Solution:

The roots of the denominator are distinct.

Hence,

$$F(s) = \frac{3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

or
$$\frac{3}{(s+2)} = A + \frac{B(s+1)}{(s+2)}$$

Let s = −1, then A = 3

Let s = −2, then

$$\frac{3}{(s+1)} = \frac{A(s+2)}{(s+1)} + B$$

or
$$\frac{3}{(-2+1)} = B$$

or
$$B = -3$$

Hence,
$$F(s) = \frac{3}{s+1} - \frac{3}{s+2}$$

From Laplace transform tables,

$$f(t) = (3e^{-t} - 3e^{-2t})\,u(t)$$

MATLAB Solution:

```
>> syms s
>> f=3/((s+1)*(s+2))
f =
3/(s+1)/(2+s)
>> pretty(ilaplace(f))

                  6 exp(- 3/2 t) sinh(1/2 t)
```

Example C.10

Find the Laplace transform F(s) of the function f(t) shown in Fig. C.10. Also, determine the limiting value of F(s) as a approaches zero.

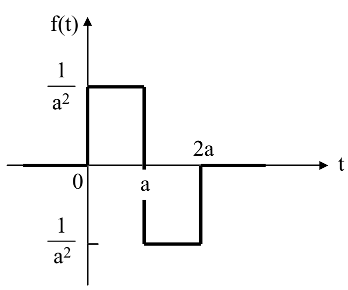

Figure C.10

Solution:

The function f(t) can be written

$$f(t) = \frac{1}{a^2}\,1(t) - \frac{2}{a^2}\,1(t-a) + \frac{1}{a^2}\,1(t-2a)$$

We have

$$
\begin{aligned}
F(s) &= L[f(t)] \\
&= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as} \\
&= \frac{1}{a^2 s}(1 - 2e^{-as} + e^{-2as})
\end{aligned}
$$

As a approaches zero

$$
\begin{aligned}
\lim_{a\to 0}(s) &= \lim_{a\to 0}\frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} = \lim_{a\to 0}\frac{\frac{d}{da}(1 - 2e^{-as} + e^{-2as})}{\frac{d}{da}(a^2 s)} \\
&= \lim_{a\to 0}\frac{2se^{-as} - 2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as} - e^{-2as}}{a} \\
&= \lim_{a\to 0}\frac{\frac{d}{da}(e^{-as} - e^{-2as})}{\frac{d}{da}(a)} = \lim_{a\to 0}\frac{-se^{-as} + 2se^{-2as}}{1} \\
&= -s + 2s = s
\end{aligned}
$$

MATLAB Solution:

```
>> f=5/((s+1)*(s+2)^2)
f =
5/(s+1)/(2+s)^2
>> pretty(ilaplace(f))

                 5 (-t - 1) exp(-2 t) + 5 exp(-t)
```

Example C.11

Find the inverse Laplace transform of

$$F(s) = \frac{5}{(s+1)(s+2)^2}$$

Solution:

The roots of $(s + 2)^2$ in the denominator are repeated. We can write

$$F(s) = \frac{5}{(s+1)(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)} \qquad (1)$$

Multiplying Eq. (1) by $(s + 1)$ on both sides and letting $s = -1$

$$
\begin{aligned}
\frac{5}{(s+2)^2} &= A + \frac{B(s+1)}{(s+2)^2} + \frac{C(s+1)}{(s+2)} \\
5 &= A
\end{aligned}
$$

Multiplying Eq. (1) by $(s + 2)^2$ and letting $s = -2$

$$\frac{5}{(s+1)} = \frac{A(s+2)^2}{(s+1)} + B + C(s+2) \tag{2}$$

or $$B = -5$$

To find C, we differentiate Eq. (2) with respect to s and letting $s = -2$

$$\frac{-5}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2}A + C$$

or $$C = -5$$

From Laplace transform tables (Table 3.1), we have

$$f(t) = 5e^{-t} - 5te^{-2t} - 5e^{-2t}$$

MATLAB Solution:

```
>>syms s;
>>F = 5/((s+1)*(s+2)^2);
>>pretty(ilaplace(F))

5 exp(-t) + 5 (-t - 1) exp(-2 t)
```

Example C.12

Determine the Laplace transform of L(sin 2t).

Solution:

We know that

$$\sin 2t = -\frac{1}{2}\frac{d(\cos 2t)}{dt} \tag{1}$$

Also $$L[\alpha f(t) + \beta g(t)] = \alpha \bar{f}(s) + \beta \bar{g}(s) \tag{2}$$

and $$L\left[\frac{d^n f}{dt^n}\right] = s^n \bar{f}(s) - s^{n-1}f(0) - s^{n-1}\bar{f}(0)\cdots$$

$$-sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{3}$$

Here, n = 1 in Eqs. (2) and (3).

Therefore,

$$L[\sin 2t] = -\frac{1}{2}(sL(\cos 2t) - 1)$$

From Laplace transform tables, we have

$$L(\sin 2t) = -\frac{1}{2}\left(\frac{s^2}{s^2+4} - 1\right) = \frac{2}{s^2+4}$$

MATLAB Solution:

```
>> syms t s a
f=(sin(2*t));
F=laplace(f,t,s)

F =

2/(s^2+4)
```

Example C.13

Find the inverse Laplace transform of

$$F(s) = \frac{4(s^2 + 5s + 2)}{s(s+1)(s+3)}$$

Solution:

$$\frac{4(s^2 + 5s + 2)}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$4(s^2 + 5s + 2) \equiv A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Substituting s = 0, −1 and −2 in the above equation, we get the values of A, B and C as 4, 8 and −8, respectively. Hence,

$$F(s) = \frac{4}{s} + \frac{8}{s+1} - \frac{8}{s+2}$$

$$f(t) = L^{-1}[F(s)] = 4L^{-1}\left[\frac{1}{s}\right] + 8L^{-1}\left[\frac{1}{s+1}\right] - 8L^{-1}\left[\frac{1}{s+2}\right]$$

From the Laplace transformation tables (Table 3.1), we obtain the solution as

$$f(t) = 4 + 8e^{-t} - 8e^{-2t}$$

MATLAB Solution:

```
>> f=4*(s^2+5*s+2)/(s*(s+1)*(s+3))
f =
(4*s^2+20*s+8)/s/(s+1)/(s+3)
>> pretty(ilaplace(f))

                  8/3 exp(-3 t) + 8/3 + 4 exp(-t)
```

Example C.14

Find the Laplace transform of f(t) defined by

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= t^2 \sin \omega t && t \geq 0 \end{aligned}$$

Solution:

We know

$$L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

Therefore

$$L[f(t)] = L[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2 + \omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2 + \omega^2)^3}$$

Example C.15

Find the inverse Laplace transform of

$$F(s) = \frac{2}{s^2(s + 4)^2}$$

Solution:

$$\frac{2}{s^2(s+4)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4} + \frac{D}{(s+4)^2}$$

$$2 \equiv As(s + 4)^2 + B(s + 4)^2 + Cs^2(s + 4) + Ds^2$$

Substituting $s = 0$ and -4 in the above equation, we get the values of B and D as

$$B = 0.125 \text{ and } D = 0.125$$

Comparing the coefficients of s^3 and s^2 terms on both sides, we get the values of A and C as

$$A = -0.0625$$

and $$C = 0.0625$$

Hence, $$\frac{2}{s^2(s+4)^2} = \frac{-0.0625}{s} + \frac{0.125}{s^2} + \frac{0.0625}{s+4} + \frac{0.125}{(s+4)^2}$$

From the Laplace transformation table (Table 3.1), we obtain the solution as

$$f(t) = -0.0625 + 0.125\, t + 0.0625\, e^{-4t} + 0.125\, t\, e^{-4t}$$

MATLAB Solution:

```
>>syms s;
>>F = 2 / (s^2*(s+4)^2);
>>pretty(ilaplace(F))

1/8 t (1 + exp(-4 t)) - 1/16 + 1/16 exp(-4 t)
```

Example C.16

Find the Laplace transform of $e^{-\xi\omega_n t} \cos\omega_d t$ where $\omega_d = \omega_n\sqrt{1 - \xi^2}$ using the first shifting theorem and the Laplace transform table.

Solution:

The first shifting theorem can be stated as

If $\qquad \bar{f}(s) = L[f(t)]$

Then $\qquad L[e^{-\alpha t}f(t)] = \bar{f}\,(s + a)$

Therefore,

$$L[e^{-\xi\omega_n t}\cos\omega_d t] = \left.\frac{s}{s^2 + \omega_d^2}\right|_{s \to s+\xi\omega_n} = \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{s + \xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Example C.17

Find the inverse Laplace transform of the function

$$F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$$

Solution:

$$L^{-1}\left[\frac{2(s+2)}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{\frac{2}{5}}{s+1} - \frac{\frac{1}{5} - \frac{3}{5}j}{s+2j} - \frac{\frac{3}{5}j}{s-2j}\right]$$

$$= \frac{2}{5}\,[e^{-t} - \cos 2t - 3j \sin 2t]\; 1(t)$$

MATLAB Solution:

```
>>syms s;
>>F = (2*(s+2)) / ( (s+1)*(s^2+4) );
>>pretty(ilaplace(F))

2/5 exp(-t) - 2/5 cos(2 t) + 6/5 sin(2 t)
```

Example C.18

Determine the Laplace transform of the function in Fig. C.18 using the second shifting theorem and the Laplace transform table.

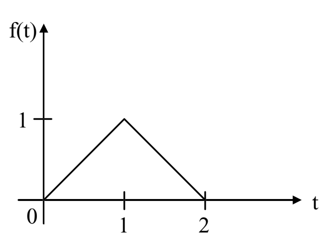

Figure C.18

Solution:

The function in Fig. C.18 can be written using unit step function as

$$\begin{aligned} f(t) &= t[u(t) - u(t-1)] + (2-t)[u(t-1) - u(t-2)] \\ &= tu(t) - 2(t-1)\,u(t-1) + (t-2)\,u(t-2) \end{aligned} \tag{1}$$

The second shifting theorem can be stated as

If $\qquad \bar{f}(s) = L[f(t)]$

Then $\qquad L[f(t-\alpha)\,u(t-\alpha)] = e^{-\alpha s}\,\bar{f}(s) \qquad (2)$

For $n = 1$ using Eq. (2) and the Laplace transform table, Eq. (1) becomes

$$L[f(t)] = \frac{1}{s} - e^{-s}\frac{2}{s} + e^{-2s}\frac{1}{s} = \frac{1}{s}(1 - 2e^{-s} + e^{-2s})$$

Example C.19

Find the inverse Laplace transform of the function

$$F(s) = \frac{s^3 + 2s + 4}{s^4 - 16}$$

Solution:

$$\begin{aligned} L^{-1}\left[\frac{s^3 + 2s + 4}{s^4 - 16}\right] &= L^{-1}\left[\frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{2}}{s-2} + \frac{\frac{1}{8} + \frac{1}{8}j}{s - 2j} + \frac{\frac{1}{8} - \frac{1}{8}j}{s - 2j}\right] \\ &= \left[\frac{1}{4}e^{-2t} + \frac{1}{2}e^{2t} + \frac{1}{8}(1+j)e^{2jt} + \frac{1}{8}(1-j)e^{-2jt}\right] 1(t) \\ &= \frac{1}{4}[e^{-2t} + 2e^{2t} + \cos 2t + j\,\sin 2t]\,1(t) \end{aligned}$$

MATLAB Solution:

```
>>syms s;
>>F = (s^3+2*s+4) /  (s^4-16) ;
>>pretty(ilaplace(F))

1/2 exp(2 t) + 1/4 exp(-2 t) + 1/4 cos(2 t) - 1/4 sin(2 t)
```

Example C.20

Determine the Laplace transform of $f_1(t) * f_2(t)$ where

$$\begin{aligned} &f_1(t) = f_2(t) = 0 && \text{for } t < 0 \\ &f_1(t) = t && \text{for } t > 0 \\ &f_2(t) = 1 + e^{-t} && \text{for } t \geq 0 \end{aligned}$$

Solution:

We know that

$$L[t] = F_1(s) = \frac{1}{s^2}$$

$$L[1 + e^{-t}] = F_2(s) = \frac{1}{s} + \frac{1}{s+1} \tag{1}$$

The Laplace transform of the convolution integral is given by

$$L[f_1(t) * f_2(t)] = F_1(s)F_2(s) = \frac{1}{s^2}\left[\frac{1}{s} + \frac{1}{s+1}\right]$$

$$= \frac{1}{s^3} + \frac{1}{s^2(s+1)} = \frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \tag{2}$$

To check this result, we can perform the integration of the convolution integral and then take its Laplace transform.

$$f_1(t) * f_2(t) = \int_0^t \tau\,[1 + e^{-(t-\tau)}]d\tau = \int_0^t (t-\tau)\,(1 + e^{-\tau})d\tau$$

$$= \frac{t^2}{2} + t - 1 + e^{-t} \tag{3}$$

$$L\left[\frac{t^2}{2} + t - 1 + e^{-t}\right] = \frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \tag{4}$$

which is identical to Eq. (2).

Example C.21

Solve the following differential equation using Laplace transforms.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = \sin t \qquad y(0) = a,\ \dot{y}(0) = b$$

Solution:

$$L\left[\frac{d^2y}{dt^2} + \frac{dy}{dt} = \sin t\right] = s^2Y(s) - sa - b + sY(s) - a = \frac{1}{s^2+1}$$

Hence,

$$Y(s) = \frac{as^3 + (a+b)s^2 + as + a + b + 1}{s(s+1)(s^2+1)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+j} + \frac{D}{s-j}$$

from which we find

$$A = a + b + 1$$
$$B = -b - \frac{1}{2}$$
$$C = \frac{1}{4}(1 + j)$$
$$D = \frac{1}{4}(1 - j)$$

Hence,

$$y(t) = \left[a + b + 1 - \left(b + \frac{1}{2}\right)e^{-t} + \frac{1}{4}(1 + j)e^{-jt} + \frac{1}{4}(1 - j)e^{jt}\right]1(t)$$
$$= \frac{1}{2}[2a + 2b + 2 - (2b + 1)e^{t} + \cos t - j\sin t]1(t)$$

MATLAB Solution:

```
>> dsolve('D2y+Dy=sin(t)','y(0)=a','Dy(0)=b')
ans =
-1/2*sin(t)-1/2*cos(t)-exp(-t)*(1/2+b)+1+a+b
```

Example C.22

Find the initial value of df(t)/dt when the Laplace transform of f(t) is given by

$$F(s) = L[f(t)] = \frac{2s + 1}{s^2 + s + 1}$$

Solution:

Using the initial value theorem,

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s(2s + 1)}{s^2 + s + 1} = 2$$

Because the L_+ transform of df(t)/dt = g(t) is

$$L_+[g(t)] = sF(s) - f(0+) = \frac{s(2s + 1)}{s^2 + s + 1} - 2 = \frac{-s - 2}{s^2 + s + 1}$$

the initial value of df(t)/dt is

$$\lim_{t\to 0+} \frac{df(t)}{dt} = g(0+) = \lim_{s\to\infty} s[sF(s) - f(0+)]$$
$$= \lim_{s\to\infty} \frac{-s^2 - 2s}{s^2 + s + 1} = -1$$

Example C.23

Solve the following differential equation using Laplace transforms.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = e^t \qquad y(0) = a, \quad \dot{y}(0) = b$$

Solution:

$$L\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = e^t\right] = s^2Y(s) - sa - b + 2sY(s) - 2a = \frac{1}{s-1}$$

or
$$Y(s) = \frac{as^2 + (a+b)s - 2a - b + 1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+2}$$

From which we get

$$A = \left.\frac{as^2 + (a+b)s - 2a - b + 1}{s+2}\right|_{s=0} = \frac{1}{2} - \frac{b}{2} - a$$

$$B = \left.\frac{as^2 + (a+b)s - 2a - b + 1}{s}\right|_{s=-2} = \frac{3b}{2} - \frac{b}{2}$$

Hence,
$$y(t) = \frac{1}{2}[1 - b - 2a + (3b - 1)e^{-2t}]1(t)$$

MATLAB Solution:

```
>> dsolve('D2y+2*Dy=exp(t)','y(0)=a','Dy(0)=b')
ans =
1/3*exp(t)-1/2*exp(-2*t)*(-1/3+b)-1/2+a+1/2*b
>> pretty(ans)

      1/3 exp(t) - 1/2 exp(-2 t) (- 1/3 + b) - 1/2 + a + 1/2 b
```

Example C.24

Find the inverse Laplace transform of

$$F(s) = \frac{(s+1)}{(s^2 + 7s + 12)}$$

Solution:

$$F(s) = \frac{(s+1)}{(s^2 + 7s + 12)} = \frac{s+1}{(s+3)(s+4)} = \frac{A_1}{s+3} + \frac{A_2}{s+4} \qquad (1)$$

where the constants A_1 and A_2 are obtained as

$$A_1 = [(s+3)F(s)]_{s=-3} = \left(\frac{s+1}{s+4}\right)_{s=-3} = -2$$

$$A_2 = [(s+4)F(s)]_{s=-4} = 3$$

Substituting into Eq. (1) gives

$$F(s) = \frac{-2}{s+3} + \frac{3}{s+4}$$

Taking the inverse Laplace transform on both sides, we obtain

$$f(t) = L^{-1}\left\{\frac{-2}{s+3}\right\} + L^{-1}\left\{\frac{3}{s+4}\right\}$$

$$= -2L^{-1}\left\{\frac{1}{s+3}\right\} + 3L^{-1}\left\{\frac{1}{s+4}\right\} = -2e^{-3t} + 3e^{-4t},\ t \geq 0$$

where, once again, the linearity of L^{-1} as been used.

MATLAB Solution:

```
>> syms F S
F=(s+1)/(s^2+7*s+12);
>> ilaplace(F)

ans =

3*exp(-4*t)-2*exp(-3*t)
```

Example C.25

Solve the second order differential equation

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 5y = 4t$$

with the boundary conditions $t = 0$; $y = 3$ and $dy/dt = -2$.

Solution:

Taking Laplace transform on both sides of the given problem using the theorems of the Laplace transform discussed in Section C.7, we get

$$\left[s^2Y(s) - sY(0) - Y'(0)\right] + 7\left[sY(s) - Y(0)\right] + 5Y(s) = 4/s^2$$

$$\left[s^2Y(s) - 3s + 2\right] + 7\left[sY(s) - 3\right] + 5Y(s) = 4/s^2$$

This can be written as

$$Y(s) = \frac{4}{s^2\left(s^2 + 7s + 5\right)} + \frac{3s + 19}{\left(s^2 + 7s + 5\right)}$$

After partial fractions as discussed in the earlier section, we get

$$Y(s) = \left[\frac{-1.12}{s} + \frac{0.8}{s^2} + 1.12\frac{s}{\left(s^2 + 7s + 5\right)} + \frac{7.04}{\left(s^2 + 7s + 5\right)}\right]$$

$$+\left[3\frac{s}{\left(s^2 + 7s + 5\right)} + \frac{19}{\left(s^2 + 7s + 5\right)}\right]$$

MATLAB Solution:

```
>> dsolve('D2y+7*Dy+5*y=4*t','y(0)=3','Dy(0)=-2')
ans =
exp(1/2*(-7+29^(1/2))*t)*(581/1450*29^(1/2)+103/50)+exp(-1-
/2*(7+29^(1/2))*t)*(-581/1450*29^(1/2)+103/50)-28/25+4/5*t
```

Example C.26

Find the inverse Laplace transform of

$$F(s) = \frac{as + b}{(s^2 + 2cs + c^2) + d^2}$$

where c and d are positive real.

Solution:

Because F(s) can be written as

$$\begin{aligned} F(s) &= \frac{a(s + c) + b - ac}{(s + c)^2 + d^2} \\ &= \frac{a(s + c)}{(s + c)^2 + d^2} + \frac{b - ac}{d} \frac{d}{(s + c)^2 + d^2} \end{aligned}$$

We get

$$f(t) = ae^{-ct} \cos dt + \frac{b - ac}{d} e^{-ct} \sin dt$$

MATLAB Solution:

```
>> syms F s a b c d
F=(a*s+b)/((s^2+2*c*s+c^2)+d^2);
ilaplace(F)

ans =

exp(-c*t)*(a*cos(d*t)+sin(d*t)*(-a*c+b)/d)
```

Example C.27

Find the Laplace transform of the function

$$\begin{aligned} f(t) &= 0 \qquad & t < 0 \\ &= te^{-13t} \qquad & t \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} f(t) &= 0 \qquad & t < 0 \\ &= te^{-13t} \qquad & t \geq 0 \end{aligned}$$

We know that $L[t] = \dfrac{1}{s^2}$

Also $$L[e^{-\alpha t} f(t)] = \int_0^{\infty} e^{-\alpha t} \; f(t) e^{-st} dt = F(s + \alpha)$$

$$F(s) = L[f(t)] = L[te^{-13t}] = \frac{1}{(s + 13)^2}$$

MATLAB Solution:

```
>> syms t
>> format compact
>> u=heaviside(t)
u =
heaviside(t)
>> f=u*t*exp(-13*t);
>> pretty(laplace(f))

          1
      ---------
         2
      (s + 13)
```

Example C.28

Find the inverse Laplace transform of

$$F(s) = \frac{3s + 5}{(s - 3)(s + 1)}$$

Solution:

We determine the constants A and B such that

$$\bar{x}(s) = \frac{3s + 5}{(s - 3)(s + 1)} = \frac{A}{s - 3} + \frac{B}{s + 1}$$

Multiplication by $(s - 3)(s + 1)$ gives

$$3s + 5 = A(s + 1) + B(s - 3)$$

or $$3s + 5 = (A + B)s + A - 3B$$

Equating similar powers of s yields

$$A = \frac{7}{2} \quad \text{and} \quad B = -\frac{1}{2}$$

Then $$\bar{x}(s) = \frac{7/2}{s - 3} - \frac{1/2}{s + 1}$$

From Table 3.1 on Laplace transformations, we have

$$x(t) = \frac{7}{2}e^{3t} - \frac{1}{2}e^{-t}$$

MATLAB Solution:

```
>> syms F s
F=(3*s+5)/((s-3)*(s+1));
>> ilaplace(F)
ans =
7/2*exp(3*t)-1/2*exp(-t)
```

Example C.29

Find the Laplace transform of the following functions.

a) $f_1(t) = 0 \qquad t < 0$
$\quad = 5\sin(3t + 45^\circ) \qquad t \geq 0$

b) $f_2(t) = 0 \qquad t < 0$
$\quad = 0.01(1 - \cos 5t) \qquad t \geq 0$

Solution:

a) $f_1(t) = 0 \qquad t < 0$
$\quad = 5\sin(3t + 45^\circ) \qquad t \geq 0$

We know that

$$\begin{aligned} 5\sin(3t + 45^\circ) &= 5\sin 3t \cos 45^\circ + 5\cos 3t \sin 45^\circ \\ &= \frac{5}{\sqrt{2}}\sin 3t + \frac{5}{\sqrt{2}}\cos 3t \end{aligned}$$

$$\begin{aligned} F1(s) = L[f_1(t)] &= \frac{5}{\sqrt{2}}\frac{3}{s^2 + 3^2} + \frac{5}{\sqrt{2}}\frac{s}{s^2 + 3^2} \\ &= \frac{5}{\sqrt{2}}\frac{s + 3}{s^2 + 9} \end{aligned}$$

MATLAB Solution:

```
>> syms u
>> f=u*5*sin(3*t+45/180*pi)
f =
5*u*sin(3*t+1/4*pi)
>> laplace(f)
ans =
5/2*u*2^(1/2)*(3+s)/(s^2+9)
```

b) $f_2(t) = 0 \qquad t < 0$
$\quad = 0.01(1 - \cos 5t) \qquad t \geq 0$

$$F_2(s) = L[f_2(t)] = 0.01\left(\frac{1}{s}\right) - 0.01\left(\frac{s}{s^2 + 5^2}\right) = \frac{0.25}{s(s^2 + 25)}$$

Example C.30

Find the inverse Laplace transform of F(s), where

$$F(s) = \frac{1}{s(s^2 + 4s + 5)}$$

Solution:

Since $$s^2 + 4s + 5 = (s + 2 + j1)\,(s + 2 - j1) \qquad (1)$$

We note that F(s) involves a pair of complex-conjugate poles, and, hence, we expand F(s) into the form

$$F(s) = \frac{1}{s(s^2 + 4s + 5)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 4s + 5} \qquad (2)$$

where a_1, a_2, and a_3 are obtained from

$$1 = a_1(s^2 + 4s + 5) + (a_2 s + a_3)s \qquad (3)$$

Comparing the coefficients of the s^2, s, and s^o terms on both sides of Eq. (3) respectively, we get

$$a_1 + a_2 = 0$$

$$4a_1 + a_3 = 0$$

and $$5a_1 = 1$$

from which

$$a_1 = 1/5$$

$$a_2 = -1/5$$

and $$a_3 = -4/5$$

Hence, $$F(s) = \frac{1}{5}\frac{1}{s} + \frac{-\frac{1}{5}s - \frac{4}{5}}{(s^2 + 4s + 5)} = \frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s + 4}{s^2 + 4s + 5}$$

$$= \frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{2}{(s + 2)^2 + 1^2} - \frac{1}{5}\frac{s + 2}{(s + 2)^2 + 1^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{1}{5} - \frac{2}{5}e^{-2t}\sin 2t - \frac{1}{5}e^{-2t}\cos t$$

MATLAB Solution:

```
>> syms F s
F=(1)/(s*(s^2+4*s+5));
>> ilaplace(F)
ans =
1/5-1/5*exp(-2*t)*(cos(t)+2*sin(t))
```

Example C.31

Find the Laplace transform of the function defined by

$$f(t) = 0 \qquad t < 0$$
$$= t^3 e^{-\alpha t} \qquad t \geq 0$$

Solution:

$$f(t) = 0 \qquad t < 0$$
$$= t^3 e^{-\alpha t} \qquad t \geq 0$$

Noting $\quad L[t^3] = \dfrac{2}{s^4}$ and referring to the equation

$$L[e^{-\alpha t} f(t)] = \int_0^\infty e^{-\alpha t}\ f(t)\ e^{-st} dt = F(s + \alpha)$$

We have $\quad F(s) = L[f(t)] = L[t^3\ e^{-\alpha t}] = \dfrac{6}{(s + \alpha)^4}$

MATLAB Solution:

```
>> syms A t
f=u*t^3*exp(-A*t)

f =

u*t^3*exp(-A*t)

>> laplace(f)

ans =

6*u/(s+A)^4
```

Example C.32

Find the inverse Laplace transform of

$$G(s) = (s - 1)/(s^2 + 2s + 2)$$

Solution:

The denominator of G(s) is an irreducible second-order polynomial and possesses a pair of simple complex conjugate roots at $s_{1,2} = -1 \pm j$. G(s) can be written as

$$G(s) = \frac{s - 1}{(s + 1 + j)(s + 1 - j)} = \frac{A}{s + 1 + j} + \frac{B}{s + 1 - j} \qquad (1)$$

where A and B, the residues, given by

$$A = [(s + 1 + j)G(s)]_{s=-1-j} = \frac{2 + j}{2j} = \frac{1}{2} - j$$

$$B = [(s + 1 - j)G(s)]_{s = -1 + j} = \frac{-2 + j}{2j} = \frac{1}{2} + j \tag{2}$$

A and B are conjugate of each other. Substitution into Eq. (1) gives

$$G(s) = \left(\frac{1}{2} - j\right)\frac{1}{s + 1 + j} + \left(\frac{1}{2} + j\right)\frac{1}{s + 1 - j}$$

Term-by-term inverse Laplace transformation gives

$$\begin{aligned} g(t) &= \left(\frac{1}{2} - j\right)e^{(-1 - j)t} + \left(\frac{1}{2} + j\right)e^{(-1 + j)t} \\ &= \left(\frac{1}{2} - j\right)e^{-t}e^{-jt} + \left(\frac{1}{2} + j\right)e^{-t}e^{-jt} \\ &= e^{-t}\left\{\left(\frac{1}{2} - j\right)(\cos t - j\sin t) + \left(\frac{1}{2} + j\right)(\cos t + j\sin t)\right\} \\ &= e^{-t}(\cos t - 2\sin t) \qquad t \geq 0 \end{aligned}$$

Example C.33

Find the Laplace transform of the function

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= \cos 3\omega t \cos 4\omega t && t \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= \cos 3\omega t \cos 4\omega t && t \geq 0 \end{aligned}$$

We know that

$$\cos 3\omega t \cos 5\omega t = \frac{1}{2}(\cos 7\omega t + \cos \omega t)$$

Therefore, we have

$$\begin{aligned} F(s) &= L[f(t)] = L\left[\frac{1}{2}(\cos 7\omega t + \cos \omega t)\right] \\ &= \frac{1}{2}\left(\frac{s}{s^2 + 49\omega^2} + \frac{s}{s^2 + \omega^2}\right) = \frac{s(s^2 + 25\omega^2)}{(s^2 + 49\omega^2)(s^2 + \omega^2)} \end{aligned}$$

MATLAB Solution:

```
>> syms w t
f=u*cos(3*w*t)*cos(4*w*t)

f =

u*cos(3*w*t)*cos(4*w*t)

>> laplace(f)

ans =

u*s*(s^2+25*w^2)/(s^2+49*w^2)/(s^2+w^2)
```

Example C.34

Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{s^2+3s+2}$$

Solution:

The partial-fraction expansion of F(s) is

$$F(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \left[(s+1)\frac{s+3}{(s+1)(s+2)}\right]_{s=-1} = \left[\frac{s+3}{s+2}\right]_{s=-1} = 2$$

$$B = \left[(s+2)\frac{s+3}{(s+1)(s+2)}\right]_{s=-2} = \left[\frac{s+3}{s+1}\right]_{s=-2} = -1$$

Hence,
$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{2}{s+1}\right] + L^{-1}\left[\frac{-1}{s+2}\right]$$

$$= 2e^{-t} - e^{-2t} \qquad t \geq 0$$

MATLAB Solution:

```
>> syms F s
F=(s+3)/(s^2+3*s+2);
>> ilaplace(F)

ans =

-exp(-2*t)+2*exp(-t)
```

Example C.35

Find the Laplace transform of the function f(t) shown in Fig. C.35.

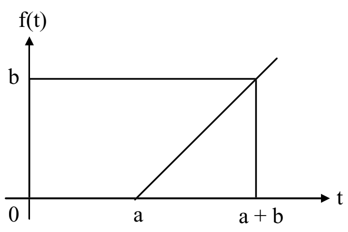

Figure C.35

Solution:

The function f(t) can be written as

$$f(t) = (t - a)\ 1(t - a)$$

The Laplace transform of f(t) is

$$F(s) = L[f(t)] = L[(t - a)\ 1(t - a)] = \frac{e^{-as}}{s^2}$$

MATLAB Solution:

```
>> syms t a
>> f=(t-a)*(t-a)

f =

(t-a)^2

>> laplace(f)

ans =

2/s^3-2*a/s^2+a^2/s
```

Example C.36

Find the inverse Laplace transform of

$$F(s) = \frac{s + 2}{(s + 3)(s + 4)}$$

Solution:

$$F(s) = \frac{s + 2}{(s + 3)(s + 4)} = \frac{A}{s + 3} + \frac{B}{s + 4} \tag{1}$$

$$s + 2 = A(s + 4) + B(s + 3) \tag{2}$$

$$s + 2 = (A + B)s + 4A + 3B \tag{3}$$

Comparing Eqs. (2) and (3), we get

$$A + B = 1 \tag{4}$$

and $$4A + 3B = 2 \tag{5}$$

Solving Eqs. (4) and (5) for A and B, we get

$$A = -1 \quad \text{and} \quad B = 2$$

Hence, $$F(s) = -\frac{1}{s+3} + \frac{2}{s+4} \tag{6}$$

From Laplace transform tables, we have

$$f(t) = -e^{-3t} + 2e^{-4t} \tag{7}$$

MATLAB Solution:

```
>> syms F s
F=(s+2)/((s+3)*(s+4));
ilaplace(F)

ans =

2*exp(-4*t)-exp(-3*t)
```

Example C.37

Find the Laplace transform of the pulse function shown in Fig. C.37.

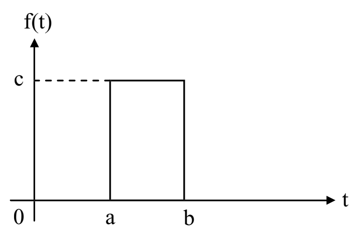

Figure C.37 Pulse function.

Solution:

The function can be written as

$$f(t) = c1(t - a) - c_1(t - b)$$

The Laplace transform of f(t) is

$$F(s) = c - \frac{e^{-as}}{s} - \frac{ce^{-bs}}{s} = \frac{c}{s}(e^{-as} - e^{-bs})$$

MATLAB Solution:

```
>> f=c*(t-a)-c*(t-b)
f =
c*(t-a)-c*(-b+t)
>> pretty(ilaplace(f))

                -c a dirac(x) + c b dirac(x)
```

Example C.38

Find the inverse Laplace transform of

$$F(s) = \frac{3(s+1)}{s^2(s+2)(s+3)}$$

Solution:

$$F(s) = \frac{3(s+1)}{s^2(s+2)(s+3)} = \frac{b_2}{s^2} + \frac{b_1}{s} + \frac{a_1}{(s+2)} + \frac{a_2}{(s+3)}$$

where

$$a_1 = \left.\frac{3(s+1)}{s^2(s+3)}\right|_{s=-2} = -3/4$$

$$a_2 = \left.\frac{3(s+1)}{s^2(s+2)}\right|_{s=-3} = 2/3$$

$$b_2 = \left.\frac{3(s+1)}{(s+2)(s+3)}\right|_{s=0} = 1/2$$

$$b_1 = \frac{d}{ds}\left[\frac{3(s+1)}{(s+2)(s+3)}\right]_{s=0}$$

$$= \left.\frac{3(s+2)(s+1) - 3(s+1)(2s+5)}{(s+2)^2(s+3)^2}\right|_{s=0} = -1/4$$

Hence,

$$F(s) = \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s} - \frac{3}{4}\frac{1}{s+2} + \frac{2}{3}\frac{1}{s+2}$$

Therefore, the inverse Laplace transform of F(s) is

$$f(t) = \frac{1}{2}t - \frac{1}{4} - \frac{3}{4}e^{-2t} + \frac{2}{3}e^{-2t} \qquad t \geq 0$$

MATLAB Solution:

```
>> syms s
>> num=3*s+3;
>> den=s^4+5*s^3+6*s^2;
>> F=num/den

F = (3*s+3)/(s^4+5*s^3+6*s^2)

>> invlaplaceF=ilaplace(F)

invlaplaceF =
         2/3*exp(-3*t)+1/12-3/4*exp(-2*t)+1/2*t

>> pretty(invlaplaceF)
           2/3 exp(-3 t) + 1/12 - 3/4 exp(-2 t) + 1/2 t
```

Example C.39

Determine the Laplace transform of the function f(t) shown in Fig. C.39. Find also the limiting value of L[f(t)] as a approaches zero.

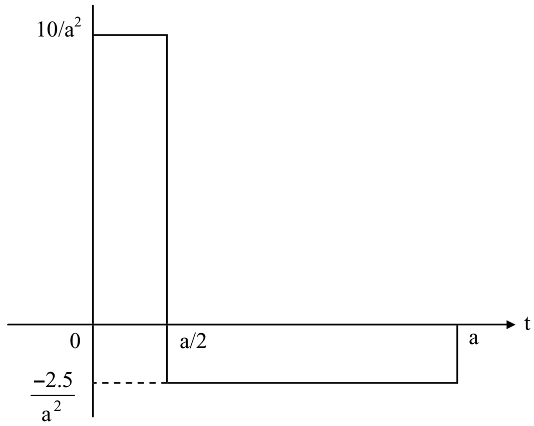

Figure C.39

Solution:

The function f(t) can be written as

$$f(t) = \frac{10}{a^2} - \frac{12.5}{a^2} 1\left(t - \frac{a}{5}\right) + \frac{2.5}{a^2} 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = L[f(t)] = \frac{10}{a^2}\frac{1}{s} - \frac{12.5}{a^2}\frac{1}{s}e^{-(a/5)s} + \frac{2.5}{a^2}\frac{1}{s}e^{-as}$$

$$= \frac{1}{a^2 s}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})$$

As a approaches zero, the limiting value of F(s) becomes as follows:

$$\lim_{a\to 0} F(s) = \lim_{a\to 0} \frac{10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as}}{a^2 s}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})}{\frac{d}{da} a^2 s}$$

$$= \lim_{a\to 0} \frac{2.5\, s\, e^{-(a/5)s} + 2.5\, s\, e^{-as}}{2as}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da} 2.5\, e^{-(a/5)s} + 2.5\, e^{-as}}{\frac{d}{da} 2a}$$

$$= \lim_{a\to 0} \frac{-0.5\, s\, e^{-(a/5)s} + 2.5\, s\, e^{-as}}{2}$$

$$= \frac{-0.5s + 2.5s}{2} = \frac{2s}{2} = s$$

MATLAB Solution:

```
>> syms t a
>> f=10/a^2-12.5/a^2*(t-a/5)+2.5/a^2*(t-a)
f =
10/a^2-25/2/a^2*(t-1/5*a)+5/2/a^2*(t-a)

>> laplace(f)

ans =
10*(s-1)/a^2/s^2
```

Example C.40

Find the inverse Laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 3}{s(s+1)}$$

Solution:

We notice that the numerator polynomial is of higher degree than the denominator polynomial. Hence, we divide the numerator by the denominator until the remainder is a fraction.

$$F(s) = s^2 + s + 2 + \frac{2s+3}{s(s+1)} = s^2 + (s+2) + \frac{a_1}{s} + \frac{a_2}{s+1}$$

where $a_1 = \left.\frac{2s+3}{s+1}\right|_{s=0} = 3$

$$a_2 = \left.\frac{2s+3}{s}\right|_{s=-1} = -1$$

Therefore

$$F(s) = s^2 + s + 2 + \frac{3}{s} - \frac{1}{s+1}$$

The inverse Laplace transform of F(s) is

$$f(t) = L^{-1}[F(s)] = \frac{d^2}{dt^2}\delta(t) + \frac{d}{dt}\delta(t) + 3 - e^{-t} \qquad t \geq 0-$$

MATLAB Solution:

```
>> syms F s
F=((s^4)*(2*s^3)*(3*s^2)*(4*s)+3)/(s*(s+1));
ilaplace(F)

ans =
24*dirac(8,t)-24*dirac(7,t)+24*dirac(6,t)-24
*dirac(5,t)+24*dirac(4,t)-24*dirac(3,t)+24*dirac(2,t)-24
*dirac(1,t)+24*dirac(t)+3-27*exp(-t)
```

Example C.41

Find the Laplace transform of the function f (t) shown in Fig. C.41. Also, find the limiting value of L [f (t)] as a approaches zero.

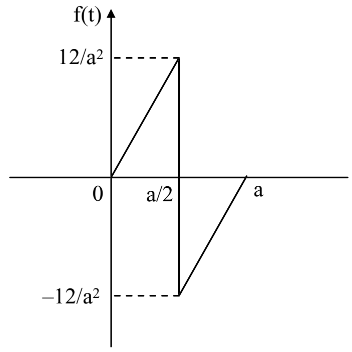

Figure C.41

Solution:

The function f(t) can be written as

$$f(t) = \frac{24}{a^3}t - \frac{24}{a^2}1\left(t - \frac{a}{2}\right) - \frac{24}{a^3}(t - a)1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \frac{24}{a^3}\frac{1}{s^2} - \frac{24}{a^2}\frac{1}{s}e^{-\frac{1}{2}as} - \frac{24}{a^3}\frac{e^{-as}}{s^2}$$

$$= \frac{24}{a^3}\left(\frac{1}{s^2} - \frac{a}{s}e^{-\frac{1}{2}as} - \frac{e^{-as}}{s^2}\right)$$

The limiting value of F(s) as a approaches zero is

$$\lim_{a\to 0} F(s) = \lim_{a\to 0}\frac{24(1 - as\,e^{-\frac{1}{2}as} - e^{-as})}{a^3s^2} = \lim_{a\to 0}\frac{\frac{d}{da}24(1 - as\,e^{-\frac{1}{2}as} - e^{-as})}{\frac{d}{da}a^3s^2}$$

$$= \lim_{a\to 0}\frac{24\left(-se^{-\frac{1}{2}as} + \frac{as^2}{2}e^{-\frac{1}{2}as} + s\,e^{-as}\right)}{3a^2s^2} = \lim_{a\to 0}\frac{\frac{d}{da}8\left(-e^{-\frac{1}{2}as} + \frac{as}{2}e^{-\frac{1}{2}as} + e^{-as}\right)}{\frac{d}{da}a^2s}$$

$$= \lim_{a\to 0}\frac{8\left[\frac{s}{2}e^{-\frac{1}{2}as} + \frac{s}{2}e^{-\frac{1}{2}as} + \frac{as}{2}\left(\frac{-s}{2}\right)e^{-\frac{1}{2}as} - se^{-as}\right]}{2as} = \lim_{a\to 0}\frac{\frac{d}{da}(4e^{-\frac{1}{2}as} - as\,e^{-\frac{1}{2}as} - 4e^{-as})}{\frac{d}{da}a}$$

$$= \lim_{a\to 0}\frac{-2s\,e^{-\frac{1}{2}as} - se^{-\frac{1}{2}as} + as\frac{s}{2}e^{-\frac{1}{2}as} + 4s\,e^{-as}}{1} = -2s - s + 4s = s$$

MATLAB Solution:

```
>> syms a t
>> f=24/a^3*t-24/a^2*(t-a/2)-24/a^3*(t-a)*(t-a)

f =

24/a^3*t-24/a^2*(t-1/2*a)-24/a^3*(t-a)^2

>> laplace(f)

ans =

-12*(-2*s-2*a*s+a^2*s^2+4)/a^3/s^3
```

Example C.42

Find the inverse Laplace transform of

$$F(s) = \frac{(s+2)}{(s+3)^2}$$

Solution:

$$F(s) = \frac{(s+2)}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

Hence, $$s + 2 = A(s + 3) + B$$

or $$s + 2 = As + 3A + B$$

Therefore, by comparing the terms, we get $A = 1$ and $3A + B = 2$ or $B = 2 - 3A = 2 - 3 = -1$

Hence, $$F(s) = \frac{1}{s+3} - \frac{1}{(s+3)^2}$$

From Laplace transform table (Table 3.1), we get

$$f(t) = e^{-3t} - te^{-3t}$$

MATLAB Solution:

```
>> syms F s
F=(s+2)/((s+3)^2);
ilaplace(F)

ans =
-(t-1)*exp(-3*t)
```

Example C.43

Given $F(s) = \dfrac{7(s+3)}{s(s+2)}$, find $f(\infty)$ using the final value theorem.

Solution:

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} \frac{s7(s+3)}{s(s+2)} = \frac{7(3)}{2} = 10.5$$

EXAMPLE C.44

Find the inverse Laplace transform of

$$F(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Solution:

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$A = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = 8/3$$

$$B = \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -3/2$$

$$C = \left.\frac{(s+2)(s+4)}{s(s+1)}\right|_{s=-3} = -1/6$$

From Laplace transform table (Table 3.1), we have

$$f(t) = \frac{8}{3}1(t) - \frac{3}{2}e^{-t}1(t) - \frac{1}{6}e^{-3t}1(t)$$

MATLAB Solution:

```
>> syms F s
F=((s+2)*(s+4))/(s*(s+3)*(s+1));
ilaplace(F)

ans =
8/3-3/2*exp(-t)-1/6*exp(-3*t)
```

EXAMPLE C.45

Given $F(s) = \dfrac{3(s+1)}{s(s+2)(s+3)}$. Obtain f(0+) using the initial value theorem.

Solution:

Using the initial value theorem

$$f(0+) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s\,3(s+1)}{s(s+2)(s+3)} = 0$$

Example C.46

Find the inverse Laplace transform of

$$F(s) = \frac{2}{(s+3)(s^2+2s+5)}$$

Solution:

$$F(s) = \frac{2}{(s+3)(s^2+2s+5)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+5}$$

where constants A, B, and C are to be determined. Combine the two fractions on the right-hand side, and equate with the expression of F(s), we have

$$\frac{2}{(s+3)(s^2+2s+5)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+5}$$
$$= \frac{A(s^2+2s+5) + (Bs+C)(s+3)}{(s+3)(s^2+2s+5)}$$

Setting the numerators equal, and collecting the like terms, we get

$$2 = (A+B)s^2 + (2A+3B+C)s + 5A + 3C$$

For this identity to be valid, the coefficients of like powers of s on both sides must be equal. Hence,

$$\begin{cases} A + B = 0 \\ 2A + 3B + C = 0 \\ 5A + 3C = 2 \end{cases}$$

Hence, $$A = \frac{1}{4} = C, \quad B = -\frac{1}{4}$$

Back substitution gives

$$F(s) = \frac{1}{4}\left\{\frac{1}{s+3} - \frac{s-1}{s^2+2s+5}\right\}$$
$$= \frac{1}{4}\left\{\frac{1}{s+3} - \frac{(s+1)-2}{(s+1)^2+2^2}\right\}$$
$$= \frac{1}{4}\left\{\frac{1}{s+3} - \frac{(s+1)}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right\}$$

The inverse Laplace transformation, using Laplace transform table (Table 3.1) gives

$$f(t) = \frac{1}{4}\{e^{-3t} - e^{-t}\cos 2t + e^{-t}\sin 2t\}$$

MATLAB Solution:

```
>> syms F s
F=2/((s+3)*(s^2+2*s+5));
ilaplace(F)

ans =

1/4*exp(-3*t)+1/4*exp(-t)*(-cos(2*t)+sin(2*t))
```

Example C.47

Find the inverse transform of the following functions.

a) $F_1(s) = \dfrac{s+3}{(s+2)(s+4)}$

b) $F_2(s) = \dfrac{3(s+1)}{s(s+2)(s+3)}$

Solution:

a) $F_1(s) = \dfrac{s+3}{(s+2)(s+4)} = \dfrac{a_1}{s+2} + \dfrac{a_2}{s+4}$

where $a_1 = \left.\dfrac{s+3}{s+4}\right|_{s=-2} = 1/2$

$a_2 = \left.\dfrac{s+3}{s+2}\right|_{s=-4} = 1/2$

$F_1(s)$ can be written as

$$F_1(s) = \frac{1}{2(s+2)} + \frac{1}{2}\left(\frac{1}{s+4}\right)$$

The inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t}$$

MATLAB Solution:

```
>> f=(s+3)/(s+2)/(s+4)
f =
(s+3)/(s+2)/(s+4)
>> pretty(ilaplace(f))

   exp(-3 t) cosh(t)
```

b) $$F_2(s) = \frac{3(s+1)}{s(s+2)(s+3)} = \frac{a_1}{s} + \frac{a_2}{s+2} + \frac{a_3}{s+4}$$

where $$a_1 = \left.\frac{3(s+1)}{(s+2)(s+4)}\right|_{s=0} = 3/8$$

$$a_2 = \left.\frac{3(s+1)}{s(s+4)}\right|_{s=-2} = +3/4$$

$$a_3 = \left.\frac{3(s+1)}{s(s+2)}\right|_{s=-4} = +9/8$$

$F_2(s)$ can be written as

$$F_2(s) = \frac{3}{8}\frac{1}{s} + \frac{3}{4}\frac{1}{s+2} + \frac{9}{8}\frac{1}{s+4}$$

The inverse Laplace transform of $F_2(s)$ is

$$f_2(s) = \frac{3}{8} + \frac{3}{4}e^{-2t} + \frac{9}{8}e^{-4t}$$

MATLAB Solution:

```
>> f=(3*s+3)/(s^2+2*s)/(s+3)
f =
(3*s+3)/(s^2+2*s)/(s+3)
>> pretty(ilaplace(f))

                    1/2 - 2 exp(-3 t) + 3/2 exp(-2 t)
```

Example C.48

Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + 3s + 5}{s^2(s^2 + 2s + 5)}$$

Solution:

$$F(s) = \frac{s^2 + 3s + 5}{s^2(s^2 + 2s + 5)} \tag{1}$$

We note that the quadratic term in the denominator involves a pair of complex conjugate roots. Hence, we can expand F(s) into the following partial-fraction form:

$$F(s) = \frac{a_1}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 5} \tag{2}$$

The coefficient a_1 is obtained as follows:

$$a_1 = \left.\frac{s^2 + 3s + 5}{s^2 + 2s + 5}\right|_{s=0} = 1$$

Hence, $$F(s) = \frac{1}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 5}$$

$$= \frac{(a_2 + b)s^3 + (1 + 2a_2 + c)s^2 + (2 + 5a_2)s + 5}{s^2(s^2 + 2s + 5)} \quad (3)$$

Now, equating the corresponding coefficients in the numerators of Eqs. (1) and (3), respectively, we get

$$\begin{aligned} a_2 + b &= 0 \\ 1 + 2a_2 + c &= 1 \\ 2 + 5a_2 &= 3 \end{aligned} \quad (4)$$

From Eq. (4), we get

$$\begin{aligned} a_2 &= 1/5 \\ b &= -1/5 \\ c &= -2/5 \end{aligned}$$

Therefore,

$$F(s) = \frac{1}{s^2} + \frac{1}{5}\frac{1}{s} + \frac{-\frac{1}{5}s - \frac{2}{5}}{s^2 + 2s + 5}$$

$$= \frac{1}{s^2} + \frac{1}{5}s + \frac{-\frac{1}{5}(s + 1) - \left(\frac{1}{5 \times 2}\right) \times 2}{(s + 1)^2 + 2^2}$$

Hence, the inverse Laplace transform of F(s) gives

$$f(t) = t + \frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t$$

MATLAB Solution:

```
>> syms F s
F=((s^2)+(3*s)+5)/(s^2*(s^2+(2*s)+5));
ilaplace(F)

ans =

1/5+t-1/10*exp(-t)*(2*cos(2*t)+sin(2*t))
```

Example C.49

Find the inverse Laplace transform of the following functions:

a) $F_1(s) = \dfrac{5s + 1}{s^2}$

b) $F_2(s) = \dfrac{6s + 3}{(s + 1)(s + 2)^2}$

Solution:

a) $F_1(s) = \dfrac{5s + 1}{s^2} = \dfrac{5}{s} + \dfrac{1}{s^2}$

The inverse Laplace transform of $F_1(s)$ is

$$f_1(s) = 5 + t$$

MATLAB Solution:

```
>> f=(5*s+1)/s^2
f =
(5*s+1)/s^2
>> pretty(ilaplace(f))

                    t + 5
```

b) $F_2(s) = \dfrac{6s + 3}{(s + 1)(s + 2)^2} = \dfrac{a}{(s + 1)} + \dfrac{b_2}{(s + 2)^2} + \dfrac{b_1}{(s + 2)}$

where $a = \left.\dfrac{6s + 3}{(s + 2)^2}\right|_{s=-1} = -3$

$$b_2 = \left.\frac{6s + 3}{(s + 1)}\right|_{s=-2} = 9$$

$$b_1 = \left.\frac{d}{ds}\left(\frac{6s + 3}{(s + 1)}\right)\right|_{s=-2} = \left.\frac{6(s + 1) - (6s + 3)}{(s + 1)^2}\right|_{s=-2}$$

$$= \frac{6(-2 + 1) - [6(-2) + 3]}{(-1)^2} = \frac{-6 + 9}{1} = 3$$

$F_2(s)$ can now be written as

$$F(s) = -3\left(\frac{1}{s + 1}\right) + 9\frac{1}{(s + 2)^2} + 3\frac{1}{(s + 2)}$$

The inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = -3e^{-t} + 9te^{-2t} + 3e^{-2t}$$

MATLAB Solution:

```
>> f=(6*s+3)/(s+1)/(s+2)^2
f =
(6*s+3)/(s+1)/(s+2)^2
>> pretty(ilaplace(f))

                   3 (3 t + 1) exp(-2 t) - 3 exp(-t)
```

Example C.50

Find the inverse Laplace transform of the function

$$F(s) = \frac{1}{(s+1)^2(s+2)}$$

Solution:

$$F(s) = \frac{1}{(s+1)^2(s+2)}$$

$$L^{-1}\left[\frac{1}{(s+1)^2(s+2)}\right] = L^{-1}\left[-\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}\right]$$

$$= -L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$= -e^{-t} + te^{-t} + e^{-2t}$$

MATLAB Solution:

```
>> F=1/((s+1)^2*(s+2));
>> ilaplace(F)

ans =

exp(-2*t)+(t-1)*exp(-t)
```

Example C.51

Find the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 3s + 3}{s(s+1)}$$

Solution:

$$F(s) = \frac{2s^2 + 3s + 3}{s(s+1)} = 2 + \frac{1}{s+1} + \frac{3}{s(s+1)}$$

$$= 2 + \frac{1}{s+1} + \frac{3}{s} - \frac{3}{s+1} = 2 - \frac{2}{s+1} + \frac{3}{s}$$

The inverse Laplace transform of F(s) is

$$f(t) = 2\delta(t) - 2e^{-t} + 3$$

MATLAB Solution:

```
>> f=(2*s^2+3*s+3)/(s*(s+1))
f =
(2*s^2+3*s+3)/s/(s+1)
>> pretty(ilaplace(f))

                   2 dirac(t) + 3 - 2 exp(-t)
```

Example C.52

Find the inverse Laplace transform of

$$F(s) = \frac{s+5}{(s+2)(s+3)}$$

Solution:

The partial-fraction expansion of F(s) is

$$F(s) = \frac{s+5}{(s+2)(s+3)} = \frac{a_1}{s+2} + \frac{a_2}{s+3}$$

where a_1 and a_2 are found from

$$a_k = \left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k}$$

$$a_1 = \left[(s+2)\frac{s+5}{(s+2)(s+3)}\right]_{s=-2} = \left[\frac{s+5}{s+3}\right]_{s=-2} = 3$$

$$a_2 = \left[(s+3)\frac{(s+5)}{(s+2)(s+3)}\right]_{s=-3} = \left[\frac{s+5}{s+2}\right]_{s=-3} = -2$$

Hence,

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{3}{s+2}\right] + L^{-1}\left[\frac{-2}{s+3}\right]$$

$$= 3e^{-2t} - 2e^{-3t} \qquad t \geq 0$$

MATLAB Solution:

```
>> syms F s
F=(s+5)/((s+2)*(s+3));
ilaplace(F)

ans =

3*exp(-2*t)-2*exp(-3*t)
```

Example C.53

Find the inverse Laplace transform of

$$F(s) = \frac{7s^2 + 5s + 8}{s^2}$$

Solution:

$$F(s) = \frac{7s^2 + 5s + 8}{s^2} = 7 + \frac{5}{s} + \frac{8}{s^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = 7\delta(t) + 5 + 8t$$

MATLAB Solution:

```
>> f=(7*s^2+5*s+8)/(s^2)
f =
(7*s^2+5*s+8)/s^2
>> pretty(ilaplace(f))

                7 dirac(t) + 5 + 8 t
```

Example C.54

Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 6s^2 + 11s + 6}{(s+2)(s+3)}$$

Solution:

We note here that the degree of the numerator polynomial is higher than that of the denominator polynomial. Therefore, we divide numerator by the denominator.

$$G(s) = (s+1) + \frac{s+5}{(s+2)(s+3)} \tag{1}$$

Now, the Laplace transform of the unit-impulse function δ(t) is 1 and the Laplace transform of dδ(t)/dt is s. The second term of Eq. (1) on the right-hand side is F(s) in C.25. Hence, the inverse Laplace transform of G(s) is given by

$$g(t) = \frac{d}{dt}(\delta(t)) + \delta(t) + 3e^{-2t} - 2e^{-3t} \qquad t \geq 0-$$

MATLAB Solution:

```
>> syms F s
F=((s^3)+(6*s^2)+(11*s)+6)/((s+2)*(s+3));
ilaplace(F)

ans =

dirac(1,t)+dirac(t)
```

Example C.55

Find the inverse Laplace transform of $F(s) = \dfrac{s-1}{s^2 + 4s + 13}$

Solution:

$$F(s) = \frac{s-1}{s^2+4s+13} = \frac{s+2-1}{s^2+4s+13}$$

$$= \frac{s+2}{(s+2)^2+3^2} - \frac{3}{(s+2)^2+3^2}\frac{1}{3}$$

Hence,
$$f(t) = e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$$

MATLAB Solution:

```
>> f=(s-1)/(s^2+4*s+13)
f =
(s-1)/(s^2+4*s+13)
>> pretty(ilaplace(f))

               exp(-2 t) cos(3 t) - exp(-2 t) sin(3 t)
```

Example C.56

Find the inverse Laplace transform of the following function

$$F(s) = \frac{s^2+9s+19}{(s^2+3s+2)(s+4)}$$

Solution:

$$F(s) = \frac{s^2+9s+19}{(s^2+3s+2)(s+4)} = \frac{s^2+9s+19}{(s+1)(s+2)(s+4)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$A = \left.\frac{s^2+9s+19}{(s+2)(s+4)}\right|_{s=-1} = 11/3$$

$$B = \left.\frac{s^2+9s+19}{(s+1)(s+4)}\right|_{s=-2} = -5/2$$

$$C = \left.\frac{s^2+9s+19}{(s+1)(s+2)}\right|_{s=-4} = -1/6$$

Hence,
$$F(s) = \frac{11}{3(s+1)} - \frac{5}{2(s+2)} - \frac{1}{6(s+4)}$$

From Laplace transform table (Table 3.1), we have

$$f(t) = \frac{11}{3}e^{-t} - \frac{5}{2}e^{-2t} - \frac{1}{6}e^{-4t}$$

MATLAB Solution:

```
>> syms F s
F=((s^2)+(9*s)+19)/(((s^2)+(3*s)+2)*(s+4));
ilaplace(F)

ans =

-1/6*exp(-4*t)-5/2*exp(-2*t)+11/3*exp(-t)
```

Example C.57

Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + 7s + 8}{s^2(s+2)}$$

Solution:

$$F(s) = \frac{s^2 + 7s + 8}{s^2(s+2)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s+2}$$

$$a = \left.\frac{s^2 + 7s + 8}{s+2}\right|_{s=0} = 4$$

$$b = \left.\frac{(s+2)(2s+7) - (s^2 + 7s + 8)}{(s+2)^2}\right|_{s=0} = 3/2$$

$$c = \left.\frac{s^2 + 7s + 8}{s^2}\right|_{s=-2} = \frac{4 - 14 + 8}{4} = \frac{-2}{4} = -1/2$$

Hence, $$F(s) = \frac{4}{s^2} + \frac{3}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$$

The inverse Laplace transform of F(s) is

$$f(t) = 4t + \frac{3}{2} - \frac{1}{2}e^{-2t}$$

MATLAB Solution:

```
>> f=(s^2+7*s+8)/(s^2*(s+2))
f =
(s^2+7*s+8)/s^2/(s+2)
>> pretty(ilaplace(f))

                 4 t + exp(-t) (cosh(t) + 2 sinh(t))
```

Example C.58

Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 6s + 13}$$

Solution:

The denominator of F(s) can be factored as

$$s^2 + 6s + 13 = (s + 3 + j2)(s + 3 - j2)$$

The roots of the denominator are complex conjugates. Therefore, we can expand F(s) into the sum of a damped sine and a damped cosine function.

Now, $s^2 + 6s + 13 = (s + 3)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$, we have

$$L[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

$$L[e^{-\alpha t} \cos \omega t] = \frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$$

Hence, F(s) can be written as a sum of a damped sine and a damped cosine function as

$$F(s) = \frac{2s + 12}{s^2 + 6s + 13} = \frac{10 + 2(s+1)}{(s+3)^2 + 2^2}$$

$$= 5\frac{2}{(s+3)^2 + 2^2} + 2\frac{s+1}{(s+3)^2 + 2^2}$$

It follows that

$$f(t) = L^{-1}[F(s)] = 5\,L^{-1}\left[\frac{2}{(s+3)^2 + 2^2}\right] + 2\,L^{-1}\left[\frac{s+1}{(s+3)^2 - 2^2}\right]$$

$$= 5e^{-3t} \sin 2t + 2e^{-3t} \cos 2t \qquad t \ge 0$$

MATLAB Solution:

```
>> syms F s
F=((2*s)+12)/((s^2)+(6*s)+13);
ilaplace(F)

ans =

exp(-3*t)*(2*cos(2*t)+3*sin(2*t))
```

Example C.59

Find the inverse Laplace transform of

$$F(s) = \frac{3s + 5}{(s+2)^2(s+3)}$$

Solution:

$$F(s) = \frac{3s + 5}{(s+2)^2(s+3)} = \frac{a}{(s+2)^2} + \frac{b}{(s+2)} + \frac{c}{s+3}$$

where
$$a = \left.\frac{3s+5}{s+3}\right|_{s=-2} = \frac{-1}{1} = -1$$

$$b = \left.\frac{(s+3)(3)-(3s+5)}{(s+3)^2}\right|_{s=-2} = \frac{3-(-1)}{1} = 4$$

$$c = \left.\frac{3s+5}{(s+2)^2}\right|_{s=-3} = \frac{-9+5}{12} = -4$$

Therefore,

$$F(s) = -1\frac{1}{(s+2)^2} + 4\frac{1}{s+2} - \frac{4}{s+3}$$

The inverse Laplace transform of F(s) is

$$f(t) = -t\,e^{-2t} + 4\,e^{-2t} - 4\,e^{-3t}$$

MATLAB Solution:

```
>> f=(3*s+5)/((s+2)^2*(s+3))
f =
(3*s+5)/(s+2)^2/(s+3)
>> pretty(ilaplace(f))
                    (4 - t) exp(-2 t) - 4 exp(-3 t)
```

Example C.60

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+\omega^2)}$$

Solution:

$$F(s) = \frac{1}{s(s^2+\omega^2)} = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2+\omega^2}\right)$$

$$= \frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2+\omega^2}$$

Therefore, the inverse Laplace transform of F(s) is given by

$$f(t) = L^{-1}[F(s)] = \frac{1}{\omega^2}(1-\cos\omega t) \qquad t \geq 0$$

MATLAB Solution:

```
>> syms F s w
F=(1)/(s*((s^2)+(w^2)));
ilaplace(F)
ans =
1/w^2*(-cos(w*t)+1)
```

Example C.61

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)}$$

Solution:

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega^2}\right)\frac{1}{\omega^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)$$

MATLAB Solution:

```
>> syms s w
>> f=1/(s^2*(s^2+w^2))

f =

1/s^2/(s^2+w^2)

>> ilaplace(f)

ans =

-1/w^3*sin(w*t)+1/w^2*t
```

Example C.62

Find the solution x(t) of the differential equation

$$\ddot{x} + 6\dot{x} + 13x = 3, \quad x(0) = 0, \quad \dot{x}(0) = 0$$

Solution:

Note that L[3] = 3/s, x(0) = 0, and $\dot{x}(0) = 0$. The Laplace transform of the differential equation becomes

$$s^2X(s) + 6sX(s) + 13X(s) = 3/s \tag{1}$$

Solving Eq. (1) for X(s), we get

$$X(s) = \frac{3}{s(s^2 + 6s + 13)} = \frac{3}{13}\frac{1}{s} - \frac{3}{13}\frac{s+6}{s^2 + 6s + 13}$$

$$= \frac{3}{13}\frac{1}{s} - \frac{3}{78}\frac{6}{(s+3)^2 + 2^2} - \frac{3}{13}\frac{s+5}{(s+3)^2 + 2^2}$$

Hence, the inverse Laplace transform is given by

$$x(t) = L^{-1}[X(s)]$$

$$= \frac{3}{13}L^{-1}[1/s] - \frac{3}{78}L^{-1}\left[\frac{6}{(s+3)^2 + 2^2}\right] - \frac{3}{13}L^{-1}\left[\frac{s+5}{(s+3)^2 + 2^2}\right]$$

or $$x(t) = \frac{3}{13} - \frac{9}{26}e^{-3t}\sin 2t - \frac{3}{13}e^{-3t}\cos 2t \quad t \geq 0$$

MATLAB Solution:

```
>> dsolve('D2x=-6*Dx-13*x+3,x(0)=0,Dx(0)=0')

ans =

-9/26*exp(-3*t)*sin(2*t)-3/13*exp(-3*t)*cos(2*t)+3/13
```

Example C.63

Find the solution x(t) of the differential equation

$$\ddot{x} + 9x = 0, \quad x(0) = 3, \quad \dot{x}(0) = 0$$

Solution:

The Laplace transform of the given differential equation is

$$[s^2X(s) - sX(0) - \dot{x}(0)] + 9X(s) = 0$$

Substituting the initial conditions, we get

$$(s^2 + 9)X(s) = 3s$$

or $$X(s) = \frac{3s}{s^2 + 9} = \frac{3s}{s^2 + 3^2}$$

The inverse Laplace transform of X(s) is

$$x(t) = 3\cos 3t$$

MATLAB Solution:

```
>> dsolve('D2x+9*x=0','x(0)=3','Dx(0)=0')
ans =
3*cos(3*t)
```

Example C.64

Find the solution of the differential equation

$$\frac{dx}{dt} + bx = A\sin\omega t \qquad x(0) = a$$

Solution:

$$\frac{dx}{dt} + bx = A\sin\omega t \qquad x(0) = a \qquad (1)$$

Taking the Laplace transform on both sides of Eq. (1), we get

$$[sX(s) - x(0)] + bX(s) = A\frac{\omega}{s^2 + \omega^2} \tag{2}$$

or

$$(s + b)X(s) = \frac{A\omega}{s^2 + \omega^2} + a \tag{3}$$

Solving Eq. (3) for X(s), we get

$$\begin{aligned} X(s) &= \frac{A\omega}{(s + b)(s^2 + \omega^2)} + \frac{a}{s + b} \\ &= \frac{A\omega}{b^2 + \omega^2}\left[\frac{1}{s + b} - \frac{s - b}{s^2 + \omega^2}\right] + \frac{a}{s + b} \\ &= \left(a + \frac{A\omega}{b^2 + \omega^2}\right)\frac{1}{s + b} + \frac{Ab}{b^2 + \omega^2}\frac{\omega^2}{s^2 + \omega^2} \\ &\quad - \frac{A\omega}{b^2 + \omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

Hence, the inverse Laplace transform of X(s) is

$$\begin{aligned} x(t) = L^{-1}[X(s)] &= \left(a + \frac{A\omega}{b^2 + \omega^2}\right)e^{-bt} + \frac{Ab}{b^2 + \omega^2}\sin \omega t \\ &\quad - \frac{A\omega}{b^2 + \omega^2}\cos \omega t \qquad t \geq 0 \end{aligned}$$

MATLAB Solution:

```
>> syms a b A w
>> dsolve('Dx=-b*x+A*sin(w*t),x(0)=a')

ans =

exp(-b*t)*(a+1/(b^2+w^2)*A*w)+A*(-cos(w*t)*w+sin(w*t)*b)/
(b^2+w^2)
```

Example C.65

Find the solution x(t) of the differential equation

$$\ddot{x} + \omega_n^2 x = t, \;\; x(0) = 0, \;\; \dot{x}(0) = 0$$

Solution:

$$\ddot{x} + \omega_n^2 x = t, \;\; x(0) = 0, \;\; \dot{x}(0) = 0$$

The Laplace transform of the differential equation is

$$s^2X(s) + \omega_n^2 X(s) = \frac{1}{s^2}$$

Solving for X(s), we get

$$X(s) = \frac{1}{s^2(s^2 + \omega_n^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega_n^2}\right)\frac{1}{\omega_n^2}$$

The inverse Laplace transform of X(s) is

$$x(t) = \frac{1}{\omega_n^2}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)$$

MATLAB Solution:

```
>> dsolve('D2x+w^2*x=t','x(0)=0','Dx(0)=0')
ans =
-sin(w*t)/w^3+1/w^2*t
```

Example C.66

Find the solution x(t) of the differential equation

$$\ddot{x} + 5\dot{x} + 6x = 0, \quad x(0) = a, \quad \dot{x}(0) = b$$

where a and b are constants.

Solution:

Writing the Laplace transform of x(t) as X(s), we have

$$L[x(t)] = X(s)$$

$$L[\dot{x}] = sX(s) - x(0) \qquad (1)$$

$$L[\ddot{x}] = s^2X(s) - sx(0) - \dot{x}(0)$$

Hence, the Laplace transform of the given differential equation is

$$[s^2X(s) - sX(0) - \dot{x}(0)] + 5[sX(s) - x(0)] + 6X(s) = 0 \qquad (2)$$

Substituting the given initial conditions in Eq. (2), we obtain

$$[s^2X(s) - as - b] + 5[sX(s) - a] + 6X(s) = 0 \qquad (3)$$

or

$$(s^2 + 5s + 6)X(s) = as + b + 5a \qquad (4)$$

Solving Eq. (4), we get

$$X(s) = \frac{as + b + 5a}{s^2 + 5s + 6} = \frac{as + b + 5a}{(s+2)(s+3)} = \frac{(3a + b)}{(s+2)} - \frac{(2a + b)}{(s+3)}$$

The inverse Laplace transform of X(s) is

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{3a + b}{s + 2}\right] - L^{-1}\left[\frac{2a + b}{s + 3}\right]$$

$$= (3a + b)e^{-2t} - (2a + b)e^{-3t} \qquad t \ge 0$$

Example C.67

Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + y = t \qquad \text{with } y(0) = 1 \text{ and } \dot{y}(0) = 0$$

Solution:

Applying the Laplace transform to the given differential equation, we get

$$s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = 1/s^2$$

Using the initial conditions, we get

$$s^2Y(s) - s + Y(s) = 1/s^2$$

or

$$Y(s) = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1}$$

Form Laplace transform Table 3.1, we obtain

$$y(t) = L^{-1}[Y(s)] = t - \sin t + \cos t$$

MATLAB Solution:

```
>> dsolve('D2y/=-y+t,y(0)=1,Dy(0)=0')

ans =

-sin(t)+cos(t)+t
```

Example C.68

Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = u(t)$$

with $y(0) = \dot{y}(0) = 0$ and $u(t) = 2\,e^{-2t}\,1(t)$

Solution:

Taking Laplace transform of both sides, we get

$$s^2Y(s) + 5sY(s) + 4Y(s) = \frac{2}{s+2}$$

or

$$Y(s) = \frac{2}{(s+2)(s+1)(s+4)}$$

The partial-fraction expansion gives

$$Y(s) = \frac{-1}{s+2} + \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

Hence,

$$y(t) = \left(-1\,e^{-2t} + \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}\right)1(t).$$

Example C.69

Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^t$$

The initial conditions are $y(0) = 0$ and $\dot{y}(0) = 2$.

Solution:

Taking Laplace transform, we get

$$L\left[\frac{d^2y}{dt^2}\right] + 4L[dy/dt] + 3L(y) = L[e^t]$$

Let $L[y](s) = Y(s)$ and with the initial conditions, we obtain

$$L\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - 2$$

and $$L\left[\frac{dy}{dt}\right] = sY(s) - y(0) = sY(s)$$

Now, $$L[e^t] = 1/s{-}1$$

Therefore,

$$s^2Y(s) - 2 + 4sY(s) + 3Y(s) = \frac{1}{s-1}$$

or $$Y(s) = \frac{2s-1}{(s-1)(s^2+4s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$

or $$A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1) = 2s-1$$

If $s = 1$, then $8A = 1$ or $A = 1/8$, for $s = -1$, $-4B = -3$ or $B = 3/4$ when $s = -3$ we get $C = -7$ or $C = -7/8$

Hence, $$Y(s) = \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+1} - \frac{7}{8}\frac{1}{s+3}$$

From Laplace transform tables, we have

$$y(t) = L^{-1}[Y(s)] = \frac{1}{8}e^t + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}$$

MATLAB Solution:

```
>> syms t
>> dsolve('D2y=-4*Dy-3*y+exp(t),y(0)=0,Dy(0)=2')

ans =

-7/8*exp(-3*t)+3/4*exp(-t)+1/8*exp(t)
```

Example C.70

Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + y = t \quad y(0) = 1 \quad \text{and} \quad \dot{y}(0) = -1$$

Solution:

$$L[\ddot{y}\,(t) + y(t) = t] \text{ gives}$$

$$s^2Y(s) - s + 1 + Y(s) = \frac{1}{s^2}$$

or

$$Y(s) = \frac{s^3 - s^2 + 1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + j} + \frac{D}{s - j}$$

we obtain

$$A = 0, \quad B = 1, \quad C = \frac{1}{2} - j \quad \text{and} \quad D = \frac{1}{2} + j$$

Hence,

$$y(t) = t + \left(\frac{1}{2} - j\right)e^{-jt} + \left(\frac{1}{2} + j\right)e^{jt}$$

$$= (t + \cos t + 2j \sin t)\ 1(t)$$

MATLAB Solution:

```
>> syms t
>> dsolve('D2y=-y+t,y(0)=1,Dy(0)=-1')

ans =

-2*sin(t)+cos(t)+t
```

Example C.71

Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0 \quad y(0) = a, \quad \dot{y}(0) = b$$

where a and b are constants.

Solution:

Let $L[y(t)] = Y(s)$

then $L[\dot{y}] = sY(s) - y(0)$

$L[\dot{y}] = s^2Y(s) - sy(0) - \dot{y}(0)$

Hence, the Laplace transform of the differential equation is

$$[s^2Y(s) - sy(0) - \dot{y}(0)] + 3[sY(s) - y(0)] + 2Y(s) = 0$$

Substituting the initial conditions, we get

$$[s^2Y(s) - as - b] + 3[sY(s) - a] + 2Y(s) = 0$$

or
$$(s^2 + 3s + 2)Y(s) = as + b + 3a$$

or
$$Y(s) = \frac{as + b + 3a}{s^2 + 3s + 2} + \frac{as + b + 3a}{(s + 1)(s + 2)}$$

$$= \frac{2a + b}{s + 1} - \frac{a + b}{s + 2}$$

Hence, the inverse Laplace transform of Y(s) gives

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{2a + b}{s + 1}\right] - L^{-1}\left[\frac{a + b}{s + 2}\right]$$

$$= (2a + b)e^{-t} - (a + b)e^{-2t} \qquad t \geq 0$$

MATLAB Solution:

```
>> syms a b
>> dsolve('D2y=-3*Dy-2*y,y(0)=a,Dy(0)=b')

ans =

(-b-a)*exp(-2*t)+(2*a+b)*exp(-t)
```

Example C.72

Find the solution x(t) of the differential equation

$$2\ddot{x} + 2\ddot{x} + x = 1 \qquad x(0) = 0, \quad \dot{x}(0) = 3$$

Solution:

$$2\ddot{x} + 2\ddot{x} + x = 1 \qquad x(0) = 0, \quad \dot{x}(0) = 3$$

The Laplace transform of this differential equation is

$$2[s^2X(s) - sX(0) - \dot{x}(0)] + 2[sX(s) - x(0)] + X(s) = \frac{1}{s}$$

Substituting the initial conditions, we get

$$2[s^2X(s) - 3] + 2[sX(s)] + X(s) = \frac{1}{s}$$

or
$$(2s^2 + 2s + 1)X(s) = 3 + \frac{1}{s}$$

Hence,
$$X(s) = \frac{3s + 1}{s(2s^2 + 2s + 1)} = \frac{3}{2s^2 + 2s + 1} + \frac{1}{s(2s^2 + 2s + 1)}$$

$$= \frac{1.5}{(s + 0.5)^2 + 0.5^2} + \frac{0.5}{s[(s + 0.5)^2 + 0.25]}$$

$$= \frac{1.5(0.5)}{(s + 0.5)^2 + 0.5^2} + \frac{1}{s} - \frac{(s + 0.5) + (0.5)}{(s + 0.5)^2 + 0.5^2}$$

MATLAB Solution:

```
>> dsolve('2*D2x=-2*Dx-x+1,x(0)=0,Dx(0)=3')

ans =

5*exp(-1/2*t)*sin(1/2*t)-exp(-1/2*t)*cos(1/2*t)+1
```

Example C.73

Find the solution x(t) of the differential equation

$$\ddot{x} + x = \sin 5t \qquad x(0) = 0, \quad \dot{x}(0) = 0$$

Solution:

The Laplace transform of the given differential equation

$$\ddot{x} + x = \sin 5t$$

is

$$s^2X(s) + X(s) = \frac{5}{s^2 + 5^2} \tag{1}$$

Solving Eq. (1) for X(s), we obtain

$$X(s) = \frac{5}{(s^2+1)(s^2+5^2)} = \frac{5}{24}\frac{1}{s^2+1} - \frac{1}{24}\frac{5}{s^2+5^2}$$

The inverse Laplace transform of X(s) gives

$$x(t) = \frac{5}{24}\sin t - \frac{1}{24}\sin 5t$$

MATLAB Solution:

```
>> syms t
>> dsolve('D2x=-x+sin(5*t),x(0)=0,Dx(0)=0')

ans =

5/24*sin(t)-1/24*sin(5*t)
```

Appendix D

Glossary of Terms

Problem D.1

Define the following terms:

a) Accident reconstruction

b) Amplitude ratio

c) Angular impulse

d) Beats

e) Braking traction coefficient

f) Centrifugal force

Solution:

a) Accident reconstruction:
Accident reconstruction is a procedure carried out with the specific purpose of estimating, in both qualitative and quantitative manner, how an accident occurred using engineering, scientific, and mathematical principles based on evidence obtained through accident investigation.

b) Amplitude ratio:
Amplitude ratio or magnification factor is the ratio of the maximum force developed in the spring of a mass-spring-dashpot system to the maximum value of the exciting force.

c) Angular impulse:
The angular impulse of a constant torque T acting for a time t is the product Tt.

d) Beats:
Beats are periodic variations that result from the superposition of two simple harmonic quantities of different frequencies.

e) Braking traction coefficient:
The maximum of the braking force coefficient can be reached without locking a wheel on a given tire and road surface for a given environmental and operating condition.

f) Centrifugal force:
Centrigugal force is the force of a body in motion that tends to keep it continuing in the same direction rather than following a curve path.

Problem D.2

Define the following terms:

a) Closing velocity

b) Collinear collision

c) Compliance steer

d) Cornering limit

e) Coulomb damping

f) Critical damping

Solution:

a) Closing velocity:
Closing velocity refers to the magnitude of the relative velocity between two vehicles at a given point in time as they approach each other; the magnitude of the relative velocity between two vehicles at the beginning of a crash; the vector difference between the velocity of the vehicle and the velocity/object struck immediately before the impact.

b) Collinear collision:
Collision collision is a collision between two vehicles whose respective directions of travel are parallel to one another, either as a rear-end or head-on collision.

c) Compliance steer:
Compliance steer is the change in steer angle of front or rear wheels resulting from compliance in suspension and steering linkages and produced by forces and/or moments applied at the tireroad contact.

d) Cornering limit:
Cornering limit is the maximum vehicle lateral acceleration of a vehicle.

e) Coulomb damping:
Coulomb damping is the damping that occurs as a result of dry friction, when two surfaces slide against each other.

f) Critical damping:
Critical damping is the minimum amount of viscous damping required in a linear system to prevent the displacement of the system from passing the equilibrium position upon returning from an initial displacement.

Problem D.3

Define the following terms:

a) Drag factor

b) Dynamic index

c) Elastic collision

d) Equivalent barrier speed

e) Flip

f) Force control

Solution:

a) Drag factor:
Drag factor is an average, uniform (constant) value of a friction coefficient applied to a specific sliding event, such as when an object slides from an initial speed to stop over a distance, d, or during a speed change, delta V.

b) Dynamic index:
(k^2/ab ratio) is the square of the radius of gyration (k) of the sprung mass about a transverse axis through the center of gravity divided by the product of the two longitudinal distances (a and b) from the center of gravity to the front and rear wheel centers.

c) Elastic collision:
Elastic collision is a collision between two bodies in which no permanent (plastic) deformation takes place and both momentum and kinetic energy are conserved.

d) Equivalent barrier speed:
Equivalent barrier speed is reached when a vehicle velocity at which the kinetic energy of the vehicle equals the energy that was absorbed in plastic (permanent) deformation.

e) Flip:
A flip is a sudden upward and downward movement off the ground when an object's horizontal movement is obstructed below its center of mass by an obstacle on the surface supporting the object. The rotation during the flip is so rapid that the object normally lands upside down. Sometimes, also called vault.

f) Force control:
Force control is the mode of vehicle control wherein inputs or restraints are placed on the steering system in the form of forces independent of the displacement required.

Problem D.4

Define the following terms:

a) Free control

b) Friction

c) Furrow

d) Simple harmonic vibration

e) Impulse

f) Jackknife

g) Lateral traction coefficient

Solution:

a) Free control:
Free control is that mode of vehicle control wherein no restraints are placed on the steering system. This is a special case of force control.

b) Friction:
Friction is the resistance to motion between two bodies in contact with each other.

c) Furrow:
A Furros is a ditch dug by a tire, wheel, or body part sliding in a dirt or loose material surface.

d) Simple harmonic vibration:
Vibration at a point in a system is simple harmonic when the displacement with respect to time is described by a simple sine function.

e) Impulse:
An impulse is a change in momentum that takes place along the principal direction of force (thrust) during a collision). Force multiplied by time.

f) Jackknife:
Jackknife refers to the behavior of a tractor-trailer system wherein the trailer center line assumes a large angle to the tractor center line.

g) Lateral traction coefficient:
The maximum value of lateral force coefficient can be reached on a free-rolling tire for a given road surface, environment, and operating condition.

Problem D.5

Define the following terms:

a) Logarithmic decrement

b) Mode of vibration

c) Normal mode of vibration

d) Overturning moment

e) Point of impact

f) Resonance

g) Roll axis

h) Roll steer

Solution:

a) Logarithmic decrement:
The rate of decay of amplitude expressed as the natural logarithm of the amplitude ratio is known as the logarithmic decrement.

b) Mode of vibration:
In a system undergoing vibration, a mode of vibration is a characteristic pattern assumed by the system in which the motion of every particle is simple harmonic with the same frequency. Two or more modes may exist concurrently in a multiple degree of freedom system.

c) Normal mode of vibration:
A normal mode of vibration is a mode of vibration that is uncoupled from (i.e., can exist independent of) other modes of vibration of a system. When vibration of the system is defined as an eigenvalue problem, the normal modes are the eigenvector and the normal mode frequencies are the eigenvalues. The term "classical normal mode" is sometimes applied to the normal modes of a vibrating system characterized by vibration of each element of the system at the same frequency and phase. In general, classical normal modes exist only in systems having no damping or having particular types of damping.

d) Overturning moment:
An overturning moment involves the component of the tire moment vector tending to rotate the tire about the X axis, positive clockwise when facing the positive direction of the X axis.

e) Point of impact:
Point of impact is the point of intersection of the contact impulse and the intravehicular contact surface during an impact.

f) Resonance:
Resonance is a forced vibration phenomenon that exists if any small change in the frequency of applied force causes a decrease in the amplitude of the vibrating system.

g) Roll axis:
Roll axis is the line joining the front and rear roll centers.

h) Roll steer:
Roll steer is the change in steer angle of front or rear wheels due to suspension roll.

Problem D.6

Define the following terms:

a) Rollover threshold

b) Scrub

c) Separation speed

d) Side slip angle

e) Sideswipe collision

f) Slip skid

g) Spring stiffness

h) Statically coupled

Solution:

a) Rollover threshold:
Rollover threshold is the lateral acceleration rate at which the vehicle will begin rotation about its longitudinal axis.

b) Scrub:
Scrub are skid marks left by tires pushed sideways or kept from rotating by forces of a collision, indicating the movement of the tire during impact between the vehicle and another vehicle or object.

c) Separation speed:
Separation speed is the speed at the time of loss of contact of two vehicles in a collision; it can refer to the speed of the centers of gravity or at the contact point.

d) Side slip angle (attitude angle):
Slip slip angle is the angle between the traces on the X-Y plane of the vehicle x-axis and the vehicle velocity vector at some specified point in the vehicle.

e) Sideswipe collision:
Sideswipe collision is refers to a collision of a vehicle during which sliding (relative tangential motion) over the intervehicular contact surface does not end at or before separation.

f) Skip skid:
A skid skid is a skid mark with short gaps, usually three feed or less from the tire bouncing off the roadway due to a hole, bump, or object. Measured as one continuous skid.

g) Spring stiffness:
A linear spring obeys a force-displacement law of F = kx, where k is called the spring stiffness or spring constant and has the dimensions of force of length, and x is the displacement of the spring.

h) Statically coupled:
If an equation of motion contains cross product of coordinates, that equation of motion is statically coupled.

Problem D.7

Define the following terms:

a) Static toe

b) Steady state vibration

c) Steering wheel torque

d) Swing center

e) Thump

f) Tractive force

g) Transmissibility

h) Vehicle wheel base

i) Yaw rate

Solution:

a) Static toe:
Static toe-in or static toe-out of a pair of wheels, at a specified wheel load or relative position of the wheel center with respect to the sprung mass, is the difference in the transverse distances between the wheel planes, taken at the extreme rear and front points of the tire treads. When the distance at the rear is greater, the wheels are "toed-in" by this amount; and where smaller, the wheels are "toed-out."

b) Steady state vibration:
Steady state vibration exists in a system if the displacement at each point recurs for equal increments of time.

c) Steering wheel torque:
Steering wheel torques is the torque applied to the steering wheel about its axis of rotation.

d) Swing center:
The instantaneous center in the transverse vertical plane through any pair of wheel centers about which the wheel moves relative to the sprung mass is the swing center.

e) Thump:
A thump is a periodic vibration and/or audible sound generated by the tire producing a pounding sensation, which is synchronous with wheel rotation.

f) Tractive force:
Tractive force is the component of the tire force vector in the direction of travel of the center of tire contact. Tractive force is equal to lateral force times sine of slip angle plus longitudinal force times cosine of slip angle.

g) Transmissibility:
Transmissibility in forced vibration is the ratio of the transmitted force to the applied force.

h) Vehicle wheel base:
   Vehicle wheel base refers to the characteristic length on which aerodynamic moment coefficients are based.

i) Yaw rate:
   Yaw rate is the angular velocity about the vertical axis.

Appendix E

Direct Numerical Integration Methods

Problem E.1

Find the response of a viscously damped single degree of freedom system subjected to a force

$$F(t) = F_o\left(1 - \sin\frac{\pi t}{t_o}\right)$$

with the following data:

$$F_O = 2N, \quad t_O = \pi, \quad M = 2.5\text{ kg}, \quad C = 0.3\text{ Ns/m}, \quad \text{and} \quad K = 1.5\text{ N/m}.$$

Assume the values of displacement and velocity of the mass at $t = 0$ to be zero. The values of the displacement and velocity of the mass at $t = 0$ are zero. Use the central difference method. Choose $\Delta t = 0.01$ s.

Solution:

This is a single degree of freedom system problem with all initial conditions zero. The following MATLAB program is executed to obtain the results.

MATLAB Program:

```
% INITIAL VALUES
m=2.5;k=1.5;c=0.3;dt=0.01;
x0=0;x0d=0;
F0=2;
T=5;
x0dd=inv(m)*(F0-c*x0d-k*x0);
xprev=x0-(dt*x0d)+((dt^2)*x0dd/2);
```

```
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=a0*m+a1*c;
   t=0;
   v(1)=x0d;a(1)=x0dd;
   i=1;
   for t=0:dt:T+dt
   X(i)=x0;
   f=F0*(1-sin(0.5*t));
fbar=f+(a2*m-k)*x0+(a1*c-a0*m)*xprev;
   x=inv(mbar)*fbar;
   xprev=x0;
   x0=x;
   i=i+1;
   p=i;
   end
   for i=2:p-1
      if i<p-1
      v(i)=(X(i+1)-X(i-1))/(2*dt);
      a(i)=(X(i+1)-2*X(i)+X(i-1))/dt^2;
      end
   end
   fprintf('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
   i=1;
   for t=0:dt:T
      fprintf('%f\t%f\t%f\t%f\n',t,X(i),v(i),a(i));
      i=i+1;
   end
   t=[0:dt:T+dt];
   plot(t,X,'-p');
xlabel('time(s)');
```

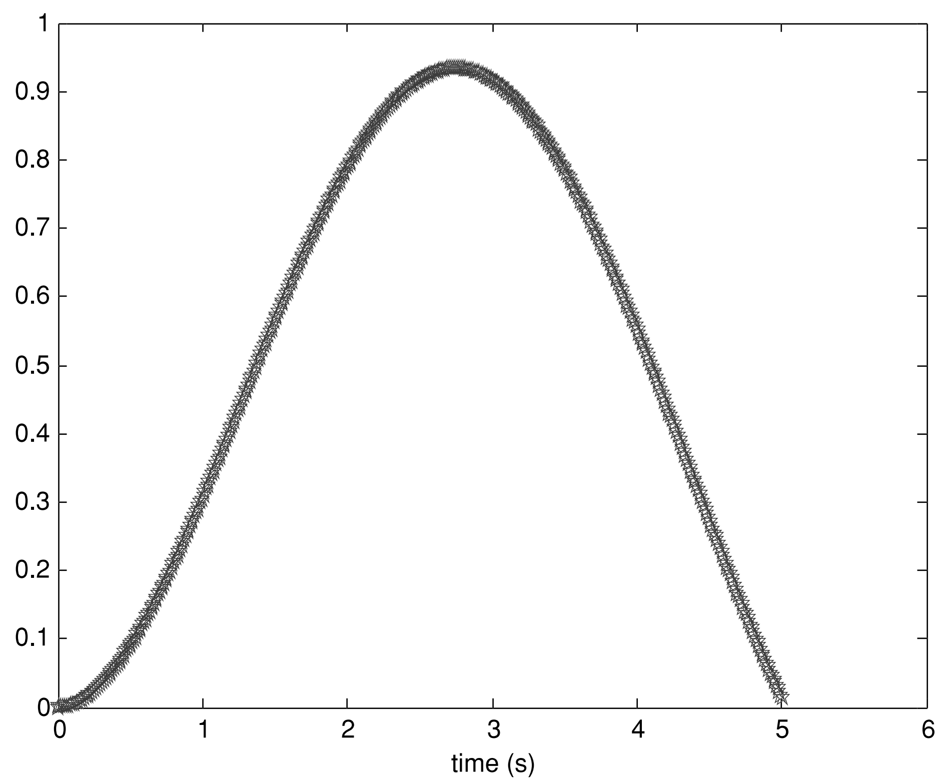

Figure E.1

PROBLEM E.2

Solve the equation of motion $m\ddot{x} + kx = F(t)$, with initial conditions $x(0) = \dot{x}(0) = 0$; the forcing function F(t) is shown in Figure E.2. Given m = 4 kg, and k = 2000 N/m. Use central difference method.

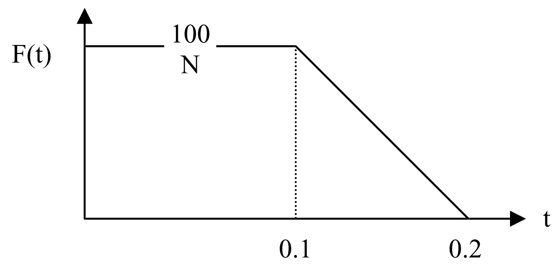

Figure E.2

Solution:

Given initial conditions are $x(0) = x_1 = \dot{x}_1 = 0$.

General central difference formula is:

$$x_{i+1} = 2x_i - x_{i-1} + h^2 \cdot f(x_i, t_i), \qquad \text{for } i \geq 2$$

For the given system, undamped natural frequency and the natural period are given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{4}} = 22.36 \text{ rad/s}$$

and period $\tau_n = \dfrac{2\pi}{\omega_n} = 0.81$ s

According to the rule $h \leq \tau_n/10$, and for convenience of representing F(t), choosing h = 0.02 s.

From the differential equation:

$$\ddot{x} = f(x, t) = \frac{1}{4}\left[F(t)\right] - 500x$$

Noting that

$$\begin{aligned} F(t) &= 100, \quad 0 \leq t \leq 0.1 \\ &= 80, \quad t = 0.12 \\ &= 60, \quad t = 0.14 \\ &= 40, \quad t = 0.16 \\ &= 20, \quad t = 0.18 \\ &= 0, \quad t \geq 0.20 \end{aligned}$$

When i = 1,

$$x_2 = x(0.02) = x_1 + h \cdot \dot{x}_1 + \frac{h^2}{2} f(x_1, t) = \frac{0.02^2}{2} \times 25 = 0.005$$

When $i \geq 2$,

$$x_3 = x(0.04) = 2x_2 - x_1 + h^2 \cdot f(x_2, t_2) = 0.005 - 0 + (0.02)^2 \cdot [25 - 500 \times 0.005]$$
$$= 0.019$$

$$x_4 = x(0.06) = 2x_3 - x_2 + h^2 \cdot f(x_2, t_2)$$
$$= 2 \times 0.019 - 0.005 + (0.02)^2 \cdot [25 - 500 \times 0.019]$$
$$= 0.0392$$

$$x_5 = x(0.08) = 2x_4 - x_3 + h^2 \cdot f(x_4, t_4) = 0.0615$$

$$x_6 = x(0.10) = 2x_5 - x_4 + h^2 \cdot f(x_5, t_5) = 0.0815$$

$$x_7 = x(0.12) = 2x_6 - x_5 + h^2 \cdot f(x_6, t_6) = 0.0932$$

The values of the response x_i obtained are shown in Table E.2.

TABLE E.2

Time (t_i)	Response (X_i)
0.00	0.000
0.02	0.005
0.04	0.019
0.06	0.0392
0.08	0.0615
0.10	0.0815
0.12	0.0932

Problem E.3

Find the free vibratory response of an undamped single degree of freedom system with $m = 1$ and $k = 1$ using the central difference method. Assuming $x_o = 0$ and $\dot{x}_o = 1$.

Solution:

The equation of motion of the system is $\ddot{x} + x = 0$. Central difference solution is $x_{i+1} = (\Delta t)^2$

$$\left\{\left(\frac{2}{(\Delta t)^2} - 1\right)x_i - \frac{1}{(\Delta t)^2}x_{i-1}\right\} = [2 - (\Delta t)^2]\, x_i - x_{i-1}$$

For $x_0 = 0$ and $\dot{x}_o = 1$, $x_{-1} = -\Delta t$

With $\Delta t = 0.5$, $x_{i+1} = 1.75\, x_i - x_{i-1}$

The results for $x_{-1} = -0.5$ and $x_0 = 0$ are shown in Table E.3.

TABLE E.3

Time (t)	x(t)
0	0
0.5	0.5
1	0.875
1.5	1.0313
2.0	0.9297
2.5	0.5957
3	0.1128
3.5	−0.3983

PROBLEM E.4

Find the free vibratory response of a viscously damped single degree of freedom system with $m = k = c = 1$, using the central difference method. Assume $x_0 = 0$, $\dot{x}_o = 1$ and $\Delta t = 0.5$.

Solution:

Here, $x_{i+1} = 0.2(7x_i - 3x_{i-1})$

$\ddot{x}_0 = -1$, $x_{-1} = -0.625$. Since $\tau_n = 2\pi/\omega_n = 2\pi$ s. See Table E.4 for the response.

TABLE E.4

i+1	x_i	x_{i-1}	x_{i+1}
1	0	−0.625	0.375
2	0.375	0	0.525
3	0.525	0.375	0.51
4	0.51	0.525	0.399
5	0.399	0.51	0.2526
6	0.2526	0.399	0.1142
7	0.1142	0.2526	0.008376
8	0.00837	0.1142	−0.0568
9	−0.0568	0.00837	−0.08457

PROBLEM E.5

Find the solution of the equation $4\ddot{x} + 2\dot{x} + 3000x = F(t)$, where F(t) is as shown in Fig. E.5 for the duration $0 \le t \le 1$. Assume zero initial conditions and $\Delta t = 0.05$.

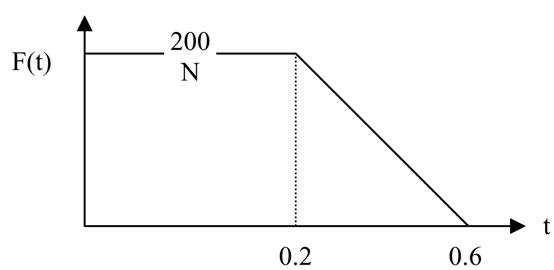

Figure E.5

Solution:

The natural frequency of the system is $\omega_n = \sqrt{\dfrac{3000}{4}} = 27.38$. The natural period is $\tau n = 2\pi/\omega_n = 0.2294$, hence, $\Delta t = 0.05$.

$$x_{i+1} = \frac{1}{1620}[200\,x_i - 1580\,x_{i-1} + F_i]$$

where

$$F(t) = 200, \quad 0 \le t \le 0.2,$$

$$= -500t + 300, \quad 0.2 \le t \le 0.6$$

$$\ddot{x}_0 = 50, \quad x_{-1} = 0.0625$$

TABLE E.5

i + 1	t_i	$F_i = F(t_i)$	x_i	x_{i-1}	x_{i+1}
1	0	200	0	0.0625	0.0625
2	0.05	200	0.0625	0	0.3111
3	0.1	200	0.3111	0.0625	0.07869
4	0.15	200	0.07869	0.1311	0.00523
5	0.2	200	0.00523	0.07869	0.04735
6	0.25	175	0.04735	0.00523	0.1087
7	0.30	150	0.1087	0.04735	0.05983
8	0.35	125	0.05983	0.1087	−0.02152
9	0.40	100	−0.0215	0.05983	0.00071
10	0.45	75	0.00071	−0.02153	0.06738
11	0.50	50	0.06738	0.000711	0.03848
12	0.55	25	0.03848	0.06738	−0.04553
13	0.6	−0.00003	−0.0455	0.03848	−0.04316

Problem E.6

Find the response of a viscously damped single degree of freedom system subjected to a force $F(t) = \left(1 - \sin\frac{t}{2}\right)$. Given m = 1 kg, c = 0.2, and k = 1. Assume the values of the displacement and velocity of the mass at t = 0 to be zero. Follow Runge-Kutta method.

Solution:

Denoting $X_1 = X$ and $X_2 = \dot{X}$, the acceleration is

$$\ddot{X} = \frac{1}{m}\left[F(t) - c\dot{X} - kX\right] = f(X, \dot{X}, t)$$

or

$$\dot{X}_1 = X_2$$
$$\dot{X}_2 = f(X_1, X_2, t)$$

Also by defining

$$\left\{X(t)\right\} = \begin{Bmatrix} X_1(t) \\ X_2(t) \end{Bmatrix}$$

and

$$\left\{F(t)\right\} = \begin{Bmatrix} X_2 \\ f(X, \dot{X}, t) \end{Bmatrix} = \begin{Bmatrix} \dot{X}(t) \\ \dfrac{1}{m}\left[(1 - \sin\dfrac{t}{2}) - c\dot{X}(t) - kX(t)\right] \end{Bmatrix}$$

Then, using recurrence formula

$$\bar{X}_{i+1} = \bar{X}_i + \frac{1}{6}\left[\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4\right]$$

with $\Delta t = \pi/10$ and the initial conditions $\left\{X(0)\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, the values of $\left\{X_{i+1}\right\}$, i = 0, 1, 2......10, obtained are shown in the Table E.6.

TABLE E.6
RESPONSE X OF THE SYSTEM OBTAINED USING THE RUNGE-KUTTA METHOD

Time (t_i)	$X_1 = X$	$X_2 = \dot{X}$
0	0	0
$\pi/10$	0.0433	0.2564
$2\pi/10$	0.1489	0.3981
$3\pi/10$	0.2826	0.4372
$4\pi/10$	0.4154	0.3977
$5\pi/10$	0.5276	0.3116
$6\pi/10$	0.6101	0.2146
$7\pi/10$	0.6649	0.1413
$8\pi/10$	0.7044	0.1207
$9\pi/10$	0.7484	0.1723
π	0.8212	0.3039

Problem E.7

Find the solution of the following equation using fourth-order Runge-Kutta method with $\Delta t = 0.05$, $m\dot{x} + kx = 0$, with $x(0) = 1$ and $\dot{x}(0) = 0$. Assume $m = k = 1$.

Solution:

Writing
$$\dot{x} = y = f(x, y, t) \tag{1}$$

The given equation becomes:

$$m\dot{y} + k\,x = 0$$

or
$$\dot{y} = -kx/m = -x = \phi(x, y, t) \tag{2}$$

With initial conditions

$$x = x_0 = 1, \qquad y = y_0 = \dot{x}_0 = 0$$

Assuming $\Delta x = k$ and $\Delta y = l$

$$x = x_o + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y = y_o + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

where

$$k_1 = h \cdot f(t_0, x_0, y_0) = 0.05 \times y_0 = 0$$

$$l_1 = h \cdot \phi(t_0, x_0, y_0) = 0.05 \times -1 = -0.05$$

$$k_2 = h \cdot f(t_0 + h/2, x_0 + k_1/2, y_0 + l_1/2) = 0.05 \times f(0.025, 1, -0.025)$$

$$= 0.05 \times -0.025 = -0.00125$$

$$l_2 = h \cdot \phi(t_0 + h/2, x_0 + k_1/2, y_0 + l_1/2) = 0.05 \times \phi(0.025, 1, -0.025)$$

$$= -0.05$$

$$k_3 = h \cdot f\,(t_0 + h/2,\, x_0 + k_2/2,\, y_0 + l_2/2) = -0.00125$$

$$l_3 = h \cdot \phi(t_0 + h/2,\, x_0 + k_2/2,\, y_0 + l_2/2) = -0.0497$$

$$k_4 = h \cdot f\,(t_0 + h,\, x_0 + k_{3,}\, y_0 + l_3) = -0.002485$$

$$l_4 = h \cdot \phi(t_0 + h/2,\, x_0 + k_{3,}\, y_0 + l_3) = -0.4993$$

Hence, $x(0.05) = 1 + \frac{1}{6}(0 + 2 \times -0.00125 + 2 \times -0.00125 - 0.002485) = 0.9991$

$$y = 1 + \frac{1}{6}(-0.05 + 2 \times -0.05 + 2 \times -0.0497 - 0.4993) = 0.8752$$

To determine y(0.1) and x(0.1) it should be considered x(0) = x(0.05) and y(0) = y(0.05). Hence, the procedure is repeated.

Problem E.8

Using the Runge-Kutta method formulates the set of first-order equations for a simple two degrees of freedom spring-mass-damper system.

Solution:

The equations of motion for the simple two degrees of freedom spring-mass-damper model are written as follows:

$$m_1\, \ddot{x}_1(t) + (c_1 + c_2)\, \dot{x}_1(t) - c_2\, \dot{x}_2(t) + (k_1 + k_2)\, x1(t) - k_2 x_2(t) = 0$$

$$m_2\, \ddot{x}_2(t) - c_2\, \dot{x}_1(t) + (c_2 + c_3)\, \dot{x}_2(t) - k_2 x_1(t) + (k_2 + k_3)\, x2(t) = 0$$

Assuming $y1 = x_1$ and $y_3 = x_2$, then

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -[(c_1 + c_2)y2 - c_2\, y_4 + (k_1 + k_2)y1 - k_2 y_3]/m_1$$

$$\dot{y}_3 = y_4$$

$$\dot{y}_4 = [c_2 y_1 - (c_2 + c_3)y4 + k_2 y_1 - (k_2 + k_3)y3]/m_2$$

With given initial conditions, the equations can be solved simultaneously. MATLAB command ode45 can be employed to solve the equations as follows:

```
t0 = [0 30];
y0 = [0.1; 0; 0.05; 0];
[t, y] = ode45 ('fun', t0, y0)
```

where '*fun*' is MATLAB dot m file name which contains the information about the function.

Problem E.9

Solve the following nonlinear vibration problem using the central difference method:

$$m\ddot{x} + c\dot{x} + k_1x + k_2x^3 = F.\ \text{cost}.$$

With $m = 1$, $c = 0.5$, $k_1 = 1$, $k_2 = 0.5$, $\Delta t = 0.5$, $t_{max} = 5.0$ and the initial conditions $x_0 = \dot{x}_0 = 0$.

Solution:

Here the in X_{i+1} an additional term with $-k_2{*}X_i^3$ will come and other things will remain same.

$$\bar{F} = F + [a_2{*}M - K]{*}X_0 + [a_1{*}C - a_0{*}M]{*}X_{prev} - k_2{*}(X_0^3);$$

MATLAB Program:

```
m=1; k=1; c=0.5; ks=0.5; dt=0.5;
x0=0; x0d=0; omega=1;
F0=10;
T=5;
x0dd=inv (m)*(F0-c*x0d-k*x0);
xprev=x0-(dt*x0d) + ((dt^2)*x0dd/2);
a0=1/dt^2; a1=1/ (2*dt); a2=2*a0;
mbar=a0*m+a1*c;
t=0;
v (1) =x0d; a (1) =x0dd;
i=1;
for t=0: dt: T+dt
X (i) =x0;
f=F0*cos (omega*t);
% NON-LINEAR TERM
fbar=f+ (a2*m-k)*x0+ (a1*c-a0*m)*xprev-ks*(x0^3);
x=inv (mbar)*fbar;
xprev=x0;
x0=x;
i=i+1;
p=i;
end
for i=2: p-1
  if i<p-1
  v (i) =(X (i+1)-X (i-1))/ (2*dt);
  a (i) =(X (i+1)-2*X (i) +X (i-1))/dt^2;
  end
end

fprintf ('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
i=1;
for t=0: dt: T
  fprintf ('%f\t%f\t%f\t%f\n', t, X (i), v (i), a (i));
  i=i+1;
end
```

TABLE E.9
RESPONSE OF THE SYSTEM OBTAINED USING THE CENTRAL DIFFERENCE METHOD WITH dt = 0.5 s AND F = 10 N

Time (t_i)	Displacement (X_i)	Velocity (V_i)	Acceleration (a_i)
0.000000	0.000000	0.000000	10.000000
0.500000	1.250000	3.677614	4.710456
1.000000	3.677614	−0.827391	−22.730475
1.500000	0.422609	−5.731781	3.112915
2.000000	−2.054167	−3.908362	4.180763
2.500000	−3.485752	1.155199	16.073482
3.000000	−0.898967	2.679238	−9.977329
3.500000	−0.806515	−1.679139	−7.456177
4.000000	−2.578107	−2.125155	5.672112
4.500000	−2.931670	2.354134	12.245043
5.000000	−0.223973	5.495064	0.318680

Problem E.10

Solve numerically the differential equation of an undamped single degree of freedom system $0.5\dot{X} + 8\pi^2 X = F(t)$, with the initial conditions $X_o = 0$ and $\dot{X}_o = 0$. Use central difference method and write a MATLAB program for a step size of 0.05 s. The forcing function is given in Fig. E.10.

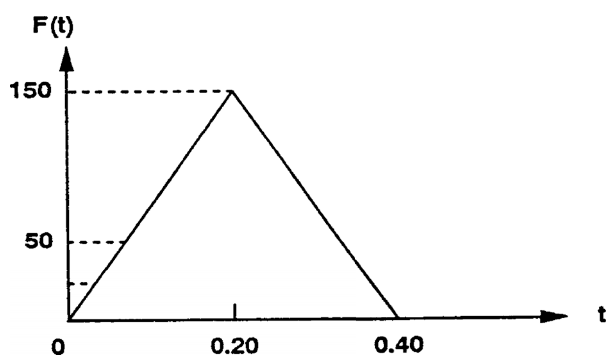

Figure E.10 Triangular wave forcing function

Solution:

Here the forcing function F(t) is defined as follows:

$$F(t) = (F*t/0.2) \quad \text{for } t <= 0.2$$
$$= -(F/0.2) \times (t - 0.4) \quad \text{for } t > 0.2 \ \& \ t <= 0.4$$
$$= 0 \quad \text{for } t > 0.4$$

MATLAB Program:

```
% INITIAL VALUES
m=0.5; k=8*pi^2; c=0; dt=0.05;
x0=0; x0d=0;
F0=0; F=150;
T=0.5;
x0dd=inv (m)*(F0-c*x0d-k*x0);
xprev=x0-(dt*x0d) + ((dt^2)*x0dd/2);
a0=1/dt^2; a1=1/ (2*dt); a2=2*a0;
```

```
mbar=a0*m+a1*c;
t=0;
v (1) =x0d; a (1) =x0dd;
i=1;
for t=0: dt: T+dt
X (i) =x0;
if t<=0.2 f= (F*t/0.2);
else if (t>0.2 & t<=0.4) f=-(F/0.2).*(t-0.4);
      else if t>0.4 f=0;
      end
    end
end
fbar=f+ (a2*m-k)*x0+ (a1*c-a0*m)*xprev;
x=inv (mbar)*fbar;
xprev=x0;
x0=x;
i=i+1;
p=i;
end
for i=2: p-1
    if i<p-1
    v (i) =(X (i+1)-X (i-1))/ (2*dt);
    a (i) =(X (i+1)-2*X (i) +X (i-1))/dt^2;
    end
end
fprintf ('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
i=1;
for t=0: dt: T
    fprintf ('%f\t%f\t%f\t%f\n', t, X (i), v (i), a (i)); i=i+1;
end
t= [0: dt: T+dt];
plot (t, X,'-p');
xlabel ('time(s)');
ylabel ('displacement (m)');
grid on;
```

The output is summarized in Table E.10.

TABLE E.10
RESPONSE OF THE SYSTEM OBTAINED USING THE CENTRAL DIFFERENCE METHOD WITH dt = 0.05 s

Time (t_i)	Displacement (X_i)	Velocity (V_i)	Acceleration (a_i)
0.000000	0.000000	0.000000	0.000000
0.050000	0.000000	1.875000	75.000000
0.100000	0.187500	6.759780	120.391187
0.150000	0.675978	12.725905	118.253838
0.200000	1.460091	17.418045	69.431745
0.250000	2.417782	15.233816	−156.800902
0.300000	2.983472	3.285518	−321.131034
0.350000	2.746334	−13.709851	−358.683716
0.400000	1.612487	−29.042788	−254.633742
0.450000	−0.157945	−34.785091	24.941603
0.500000	−1.866022	−26.794791	294.670399

Problem E.11

Find the response of the two degrees of freedom system when $F_1(t) = 0$ and $F_2(t) = 10$ using the central difference method. The mass, stiffness, and damping matrices for this system are given as

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad [K] = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}, \quad [C] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

All the initial conditions are given as zero and $\Delta t = 0.2421$ s.

Solution:

The equations of motion are given by

$$[M]\ddot{X} + [C]\dot{X} + [K]X = F(t)$$

with

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$$

$$F(t) = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

The undamped natural frequencies and the mode shapes of the system can be found by solving the eigenvalue problem:

$$\left[\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The solution of above equation is given by

$$\omega_1 = 1.807 \quad \text{and} \quad \omega_2 = 2.594$$

Correspondingly,

$$U^{(1)} = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix} \quad \text{and} \quad U^{(2)} = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}$$

Hence, the actual (natural) periods of the system are

$$\tau_1 = \frac{2\pi}{\omega_1} = 3.475 \quad \text{and} \quad \tau_2 = \frac{2\pi}{\omega_2} = 2.421$$

Selecting time step (Δt) as $\tau_2/10 = 0.2421$, the initial value of acceleration is

$$\ddot{X}_0 = [M]^{-1}\{F - [K]\,X0\}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

For every time-step, the value of X_{-1} as follows:

$$X_{-\Delta t} = X_0 - \Delta t\,\dot{X}_0 + \frac{\Delta t^2}{2}\ddot{X}_0$$

$$= \begin{Bmatrix} 0 \\ 0.1466 \end{Bmatrix}$$

$$\hat{f}(t) = F(t) - \left(K - \frac{2}{\Delta t^2}M\right)X_t - \left(\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right)X_{t-\Delta t}$$

$$= \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} - \left(\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix} - \frac{2}{0.2421^2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)X_t - \frac{1}{0.2421^2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}X_{t-\Delta t}$$

and

$$X_{t+\Delta t} = \left[\frac{1}{\Delta t^2}[M] + \frac{1}{2\Delta t}[C]\right]^{-1}\hat{f}(t)$$

$$= \left[\frac{1}{0.2421^2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right]^{-1}\hat{f}(t)$$

$$= \begin{bmatrix} 17.061 & 0 \\ 0 & 8.53 \end{bmatrix}\hat{f}(t)$$

Now these equations can be applied recursively to obtain X_1, X_2,… The results are shown in Table E.11.

TABLE E.11

Time $t_i = i \cdot \Delta t$	X: x_1	X: x_2
0	0	0
0.2421	0	0.1466
0.4842	0.0172	0.552
0.7263	0.0931	1.122
0.9684	0.267	1.727
1.2105	0.551	2.237
1.4526	0.9027	2.547
1.6947	1.235	2.605
1.9368	1.439	2.418
2.1789	1.4202	2.042
2.421	1.141	1.563
2.6631	0.6437	1.0773
2.9053	0.0463	0.669

Problem E.12

Calculate the response of a simple two degrees of freedom system given below by the central difference method.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \tag{E.1}$$

Assume the initial conditions as $X_1 = X_2 = \dot{X}_1 = \dot{X}_2 = 0$ at $t = 0$

Solution:

To find the natural frequencies, we set the determinant of the coefficient matrix $\{X\}$ equal to zero. That is,

$$\left[[K] - \omega^2[M]\right] \{X\} = \tag{E.2}$$

$$\begin{vmatrix} 4 - 2\lambda & 5 \\ 2 & 6 - 2\lambda \end{vmatrix} = 0 \tag{E.3}$$

The expansion of the determinant in Eq. (E.2) leads to

$$4\lambda^2 - 20\lambda + 14 = 0 \tag{E.4}$$

By setting $\lambda = \omega^2$, we obtain λ_1 and λ_2 from Eq. (E.3) as

$$\lambda_1 = \omega_1^2 = 4.16 \quad \text{and} \quad \lambda_2 = \omega_2^2 = 0$$

Then, the periods of oscillation are

$$\tau_1 = 2\pi/\omega_1 = 3.08 \quad \text{and} \quad \tau_2 = 2\pi/\omega_2 = 6.85$$

Let us choose the time step $\Delta t = (\tau/10) = 0.308$ to calculate the response of the system for 10 steps. The first step is to calculate $\{\ddot{X}_o\}$ using the given equations of motion at time 0; i.e., we use

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \{\ddot{X}_o\} + \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} [X_0] = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Hence,

$$\{\ddot{X}_o\} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

Now, we have

$$a_o = \frac{1}{(\Delta t)^2} = 10.54$$

$$a_1 = \frac{1}{2(0.308)} = 1.623$$

$$a_2 = 2a_o = 21.08$$

$$a_3 = \frac{1}{a_2} = 0.0474$$

Hence, at t = 0

$$\{X_{t-\Delta t}\} = \{X_o\} - \Delta t\{\dot{X}_o\} + a_3\{\ddot{X}_o\}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.308\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.0474\begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 0.1185 \\ 0.1185 \end{bmatrix}$$

$$[\bar{M}] = a_o[M] + a_1[C] = 10.54\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + 1.623\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 21.08 & 0 \\ 0 & 21.08 \end{bmatrix}$$

The effective force vector at time t is

$$\{\bar{F}_t\} = \{F_t\} - ([K] - a_2[M])\{X_t\} - (a_o[M] - a_1[C])\{X_{t-\Delta t}\} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$-\left(\begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} - 21.08\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-\left(10.54\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - 1.623\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0.1185 \\ 0.1185 \end{bmatrix} = \begin{bmatrix} 2.502 \\ 2.502 \end{bmatrix}$$

Hence, we need to solve the following equations for each time step.

$$[\bar{M}]\ \{X_{t+\Delta t}\} = \{\bar{F}_t\} \tag{E.5}$$

The solution obtained for 10 time steps using Eq. (E.5) are summarized in Table E.12.

TABLE E.12

Time	**Δt**	**2Δt**	**3Δt**	**4Δt**	**5Δt**
X_1	0.1186	0.440	0.877	1.316	1.661
X_2	0.1186	0.412	0.722	0.855	0.669

Time	**6Δt**	**7Δt**	**8Δt**	**9Δt**	**10Δt**
X_1	1.865	1.939	1.943	1.954	2.029
X_2	0.137	−0.639	−1.457	−2.083	−2.342

It is summarized with following steps:

Since [C] is zero, we have:

$$\frac{1}{\Delta t^2}[M]U_{n+1} = R_n - \left([K] - \frac{2}{\Delta t^2}[M]U_n - \frac{1}{\Delta t^2}[M]U_{n-1}\right)$$

The initial conditions could be rewritten as:

$$U_0 = 0, \quad \dot{U}_0 = 0, \quad \ddot{U}_0 = [M]^{-1}(R_0 - [K]U_0).$$

And we have:

$$U_1 = U_0 + \Delta t\,\dot{U}_0 + \frac{1}{2}\Delta t^2\,\ddot{U}_0$$

So this method could be applied beginning from n = 1.

MATLAB Program:

```
  M=[2 0;0 2];
  K=[4 5;2 6];
  C=[0.0 0.0;0.0 0.0];
dt=0.308;
x0=[0;0];x0d=[0;0];
F0=[5;5];
T=10;
x0dd=inv(M)*(F0-C*x0d-K*x0);
xprev=x0-(dt.*x0d)+((dt^2).*(x0dd/2));
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=(a0.*M)+(a1.*C);
t=0;
v(:,1)=x0d;a(:,1)=x0dd;
i=1;
fprintf('time\t\tX(1)\t\tX(2)\n');
for t=0:dt:T+dt
X(:,i)=x0;
F=F0;
Fbar=F+(a2.*M-K)*x0+(a1.*C-a0.*M)*xprev;
x=inv(mbar)*Fbar;
xprev=x0;
x0=x;
fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
i=i+1;
p=i;
end
for i=2:p-1
if i<p-1
v(:,i)=(X(:,i+1)-X(:,i-1)).*(1/(2*dt));
a(:,i)=(X(:,i+1)-2*X(:,i)+X(:,i-1)).*(1/dt^2);
end
end
```

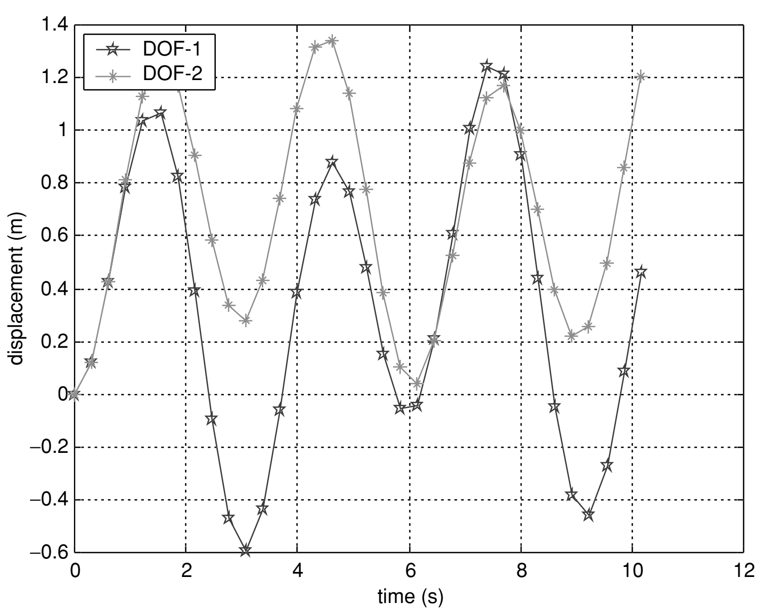

Figure E.12

Problem E.13

Find the response of the two degrees of freedom system when $F_1(t) = 0$ and $F_2(t) = 0$, using the Wilson-theta method. The mass, stiffness, and damping matrices for this system are given as

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [K] = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}, \quad [C] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Assuming initial conditions to be

$$x_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \dot{x}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \Delta t = 0.25 \text{ s.}$$

Solution:

Natural frequencies of the system are given by

$$(2 - \omega^2)(5 - \omega^2) - 4 = 0 \quad \text{or} \quad \omega^2 = 1 \text{ or } 6.$$

Hence, periods of oscillation are 2π and $2\pi/\sqrt{6}$. Let $\Delta t = 0.25$ s, which is less than $1/10^{th}$ of second period of oscillation. Also

$$\{\ddot{X}_0\} = -[K]\{X_0\} = \begin{Bmatrix} -2 \\ -2 \end{Bmatrix}$$

Assuming $\theta = 1.4$,

$$a_1 = \frac{6}{(\theta\Delta t)^2} = \frac{6}{(1.4 \times 0.25)^2} = 48.98$$

$$a_2 = \frac{3}{\theta\Delta t} = \frac{3}{(1.4 \times 0.25)} = 8.57$$

$$a_3 = \frac{6}{\theta\Delta t} = \frac{6}{1.4 \times 0.25} = 17.14$$

$$a_4 = \frac{\theta\Delta t}{2} = \frac{1.4 \times 0.25}{2} = 0.175$$

Now, the effective stiffness matrix is:

$$[\hat{K}] = [K] + a_1[M] + a_2[C] = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} + 48.98\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 50.98 & 2 \\ 2 & 53.98 \end{bmatrix}$$

For each step:

$$\hat{F}_{t+\theta\Delta t} = F_t + \theta(F_{t+\Delta t} - F_t) + [M](a_1X_t + a_3\dot{X}_t + 2\ddot{X}_t) + [C](a_2X_t + 2\dot{X}_t + a_4\dot{X}_t)$$

$$= [M](48.98\,X_t + 17.14\dot{X}_t + 2\ddot{X}_t)$$

$$\ddot{X}_{t+\Delta t} = \frac{a_1}{\theta}(X_{t+\theta\Delta t} - X_t) - \frac{a_3}{\theta}\dot{X}_t + \left(1 - \frac{3}{\theta}\right)\ddot{X}_t$$

$$= 34.98(X_{t+\theta\Delta t} - X_t) - 12.26\dot{X}_t - 1.14\ddot{X}_t$$

$$\dot{X}_{t+\Delta t} = \dot{X}_t + \frac{\Delta t}{2}(\ddot{X}_{t+\Delta t} + \ddot{X}_t) = \dot{X}_t + 0.125(\ddot{X}_{t+\Delta t} + \ddot{X}_t)$$

and

$$X_{t+\Delta t} = \left[X_t + \Delta t\,\dot{X}_t + \frac{\Delta t^2}{6}(\ddot{X}_{t+\Delta t} + 2\ddot{X}_t)\right]$$

$$= X_t + 0.25\,\dot{X}_t + 0.0104(\ddot{X}_{t+\Delta t} + 2\ddot{X}_t)$$

Here, displacement vector is obtained by solving the equation

$$\hat{K}X_{t+\theta\Delta t} = \{\hat{F}_{t+\theta\Delta t}\}$$

The computed values of X are given in Table E.13.

TABLE E.13

Time $t_i = i \cdot \Delta t$	X	
	x_1	x_2
0	1	0
0.25	0.9397	–0.0566
0.5	0.771	–0.204
0.75	0.526	–0.374
1.0	0.268	–0.512
1.25	0.052	–0.512
1.5	–0.092	–0.381
1.75	–0.235	–0.112
2.00	–0.352	0.237
2.25	–0.492	0.259

Problem E.14

Compute the time response of the system given by following equations, using Runge-Kutta method

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$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t$$

subject to initial conditions:

$$X_0 = \{0 \quad 0.1\}^T \text{ m} \quad \text{and} \quad V_0 = [1 \quad 0]^T \text{ m/s.}$$

Solution:

Given equations can be expanded as:

$$2\ddot{x}_1 + 3\dot{x}_1 - 0.5\dot{x}_2 + 3x_1 - x_2 = \sin 2t$$

$$\ddot{x}_2 - 0.5\dot{x}_1 + 0.5\dot{x}_2 - x_1 + x_2 = \sin 2t$$

Substituting

$$\dot{x}_1 = y_1 = f_1(x_1, x_2, y_1, y_2, t) \quad \text{and} \quad \dot{x}_2 = y_2 = f_2(x_1, x_2, y_1, y_2, t) \tag{1}$$

we get

$$2\dot{y}_1 + 3y_1 - 0.5y_2 + 3x_1 - x_2 = \sin 2t \tag{2}$$

or

$$\dot{y}_1 = \phi_1(x_1, x_2, y_1, y_2, t)$$

and

$$\dot{y}_2 - 0.5y_1 + 0.5y_2 - x_1 + x_2 = \sin 2t \tag{3}$$

or

$$\dot{y}_2 = \phi_2(x_1, x_2, y_1, y_2, t)$$

Solving these first-order equations using Runge-Kutta's method with initial conditions:

$$x_{01} = 0, \; x_{02} = 0.1 \text{ m}$$

and

$$\dot{x}_{01} = 1 \quad \text{and} \quad \dot{x}_{02} = 0 \text{ m/s}$$

The final values for a time range of 10 s are given in Table E.14.

TABLE E.14

t	x_1	x_2	$\dot{x}_1$	$\dot{x}_1$
0	0	0.1	1	0
1	0.455	0.25	0.079	0.425
2	0.547	0.985	–0.0118	0.649
3	0.349	0.905	–0.277	0.651
4	0.0705	0.0604	–0.171	–0.725
5	0.0888	–0.0464	0.0321	0.199
6	–0.0183	–0.0985	–0.195	–0.279
7	–0.233	–0.53	–0.088	–0.277
8	–0.0785	–0.186	0.225	0.655
9	0.0408	0.227	–0.00143	0.0825
10	–0.0839	–0.109	–0.0781	–0.401

Problem E.15

Solve Problem E.13 using Newmark's method alpha = 0.25 and beta = 0.5.

Solution:

Assuming $\Delta t = 0.25$ s, one can obtain

$$\ddot{X}_0 = \begin{Bmatrix} -2 \\ -2 \end{Bmatrix}$$

Choose $\alpha = 0.25$, $\partial = 0.5$ the constants are

$$a_1 = \frac{1}{\alpha \cdot \Delta t^2} = 64,\ a_2 = \frac{\partial}{\alpha \cdot \Delta t} = 8,\ a_3 = \frac{1}{\alpha \cdot \Delta t} = 16,\ a_4 = \frac{1}{2\alpha} - 1$$

$$a_5 = \frac{\partial}{\alpha} - 1 = 1,\ a_6 = \frac{\Delta t}{2}\left(\frac{\partial}{\alpha} - 2\right) = 0,\ a_7 = \Delta t(1 - \partial) = 0.125,\ a_8 = \partial \cdot \Delta t = 0.125$$

$$\therefore [\hat{K}] = [K] + a_1[M] = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} + 64\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 66 & 2 \\ 2 & 69 \end{bmatrix}$$

$$\hat{F}_{t+\Delta t} = [M](64X_t + 16\dot{X}_t + \ddot{X}_t) = 64X_t + 16\dot{X}_t + \ddot{X}_t$$

Because [M] is identify matrix.

Solve $X_{t+\Delta t} = [\hat{K}]^{-1} \cdot \hat{F}_{t+\Delta t}$

$$\ddot{X}_{t+\Delta t} = 64(X_{t+\Delta t} - X_t) - 16 \cdot \dot{X}_t - \ddot{X}_t$$

$$\dot{X}_{t+\Delta t} = \dot{X}_t + 0.125\ddot{X}_t + 0.125\ddot{X}_{t+\Delta t}$$

The values of {X} computed for various t are shown in Table E.15.

TABLE E.15

	X	
Time = i · Δt	**x_1**	**x_2**
0	1	0
0.25	0.939	−0.056
0.5	0.7699	−0.202
0.75	0.529	−0.375
1.0	0.264	−0.497
1.25	0.0217	−0.504
1.5	0.136	−0.363
1.75	−0.243	−0.107
2.00	−0.373	0.242
2.25	−0.499	0.271

It is important to note that unless ∂ is taken as 1/2, there is a spurious damping introduced proportional to $(\partial - 1/2)$. If ∂ is taken as zero, a negative damping results; this involves a self-excited vibration arising due to numerical procedure. Similarly, if ∂ is greater then 1/2, a positive damping is introduced. This reduces the magnitude of response even without real damping in the problem. The method is unconditionally stable for $\alpha > \frac{1}{2}\left(\partial + \frac{1}{2}\right)^2$ and $\partial \geq 1/2$.

Note: For the purpose of computation, a continuous system is idealized as a multi-degree of freedom system (say n-degrees of freedom). The response consists of all the n-modes of vibration. But generally, the first m-modes determine mostly the overall response. If the natural period of the m^{th} mode is T_m, choice of Δt equal to $T_m/10$ gives a reasonable dynamic response up to the m^{th} mode. This also represents lower mode responses accurately.

For the central difference scheme, Δt should be smaller than the natural period of practically all the n-modes to obtain reliable results.

PROBLEM E.16

Determine the displacement response of a simple two degrees of freedom system given below by analytical method.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Assume the initial conditions as $X_1 = X_2 = \dot{X}_1 = \dot{X}_2 = 0$ at $t = 0$. Compute the response of the system using time step $\Delta t = 0.308$ s for 10 equal steps.

Solution:

The response $\{X(t)\}$ of the given equations of motion consists of two parts: first, $\{X_H(t)\}$ is the transient response (i.e., the homogeneous or complementary solutions of the equations of motion); and second, $\{X_P(t)\}$ is the steady state or forced response (i.e., the particular solution to constant inputs are constants, say Z_1, Z_2). Hence,

$$\begin{aligned} 4Z_1 + 5Z_2 &= 5 \\ 2Z_1 + 6Z_2 &= 5 \end{aligned} \tag{E.1}$$

or $$Z_1 = 5/14 \quad \text{and} \quad Z_2 = 5/7$$

The complementary solution to the equations is obtained from

$$\begin{aligned} 2\ddot{X}_1 + 4X_1 + 5X_2 &= 0 \\ 2\ddot{X}_2 + 2X_1 + 6X_2 &= 0 \end{aligned} \tag{E.2}$$

Because there is no damping in the system, $\{X_H(t)\}$ oscillate at the same frequency and are either in phase or 180 degrees out of phase. Let $X_1 = x_1 e^{i\omega t}$ and $X_2 = x_2 e^{i\omega t}$ and substituting these in Eq. (E.2), we get

$$\begin{aligned} \left(4 - 2\omega^2\right)x_1 + 5x_2 &= 0 \\ 2x_1 + \left(6 - 2\omega^2\right)x_2 &= 0 \end{aligned} \tag{E.3}$$

Solving Eq. (E.3) for x_1/x_2 gives

$$\frac{x_1}{x_2} = \frac{5}{2\omega^2 - 4} = \frac{2\omega^2 - 6}{2} \tag{E.4}$$

From Eq. (E.4), the frequency equation is

$$2(\omega^2)^2 - 10(\omega^2) + 7 = 0$$

from which

$$\omega_1^2 = \frac{1}{2}(5 - \sqrt{11}) \quad \text{and} \quad \omega_2^2 = \frac{1}{2}(5 + \sqrt{11})$$

At $\quad \omega_1^2 = \frac{1}{2}(5 - \sqrt{11}) \quad \left(\frac{x_1}{x_2}\right)^{\mathrm{I}} = \frac{5}{1 - \sqrt{11}} = \lambda_1$

At $\quad \omega_2^2 = \frac{1}{2}(5 + \sqrt{11}) \quad \left(\frac{x_1}{x_2}\right)^{\mathrm{II}} = \frac{5}{1 + \sqrt{11}} = \lambda_2$

The two modes of vibration are

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{\mathrm{I}} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{\mathrm{I}} \sin(\omega_1 t + \varphi_1)$$

and

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{\mathrm{II}} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{\mathrm{II}} \sin(\omega_2 t + \varphi_2)$$

where $\frac{x_1}{x_2}$ is specified for each mode. The general solution is

$$\begin{aligned} X_1 &= Z_1 + x_1^{\mathrm{I}}\sin(\omega_1 t + \varphi_1) + x_1^{\mathrm{II}}\sin(\omega_2 t + \varphi_2) \\ X_2 &= Z_2 + x_2^{\mathrm{I}}\sin(\omega_1 t + \varphi_1) + x_1^{\mathrm{II}}\sin(\omega_2 t + \varphi_2) \end{aligned} \tag{E.5}$$

where $x_1^{\mathrm{I}}, x_2^{\mathrm{I}}, x_1^{\mathrm{II}}, x_2^{\mathrm{II}}, \varphi_1$ and φ_2 are determined from initial conditions.

At $t = 0$, $x_1 = x_2 = \dot{x}_1 = \dot{x}_2 = 0$. Substituting these and Z_1 and Z_2 in Eqs. (E.5), we get

$$\begin{aligned} 0 &= \frac{5}{14} + x_1^{\mathrm{I}}\sin\varphi_1 + x_1^{\mathrm{II}}\sin\varphi_2 \\ 0 &= \frac{5}{7} + x_2^{\mathrm{I}}\sin\varphi_1 + x_2^{\mathrm{II}}\sin\varphi_2 \\ 0 &= \omega_1 x_1^{\mathrm{I}}\cos\varphi_1 + \omega_2 x_1^{\mathrm{II}}\cos\varphi_2 \\ 0 &= \omega_2 x_2^{\mathrm{I}}\cos\varphi_1 + \omega_2 x_2^{\mathrm{II}}\cos\varphi_2 \end{aligned} \tag{E.6}$$

The last of the two equations in (E.6) gives $\cos\varphi_1 = 0$ and $\cos\varphi_2 = 0$

This is $\sin\varphi_1 = 1$ and $\sin\varphi_2 = 1$. The first of the two equations in (E.6) gives

$$\begin{aligned} \lambda_1 x_2^{\mathrm{I}} + \lambda_2 x_2^{\mathrm{II}} + \frac{5}{14} &= 0 \\ x_2^{\mathrm{I}} + x_2^{\mathrm{II}} + \frac{5}{7} &= 0 \end{aligned} \tag{E.7}$$

Solving Eq. (E.7) gives

$$x_2^{I} = \frac{5}{154}\left[2\sqrt{11} - 11\right]$$

$$x_2^{II} = -\frac{5}{154}\left[2\sqrt{11} - 11\right]$$

$$x_1^{I} = \frac{25}{154}\left[\frac{(2\sqrt{11} - 11)}{-(1 - \sqrt{11})}\right]$$

Therefore,

$$x_1^{II} = -\frac{25}{154}\left[\frac{(2\sqrt{11} + 11)}{-(1 + \sqrt{11})}\right]$$

Therefore, the response of the system is given by

$$X_1 = \frac{5}{14} + \frac{25}{154}\left(\frac{(2\sqrt{11} - 11)}{(\sqrt{11} - 1)}\right)\cos\omega_1 t - \frac{25}{154}\left(\frac{(2\sqrt{11} + 11)}{(\sqrt{11} + 1)}\right)\cos\omega_2 t$$

$$X_2 = \frac{5}{14} + \frac{5}{154}(2\sqrt{11} - 11)\cos\omega_1 t - \frac{5}{154}(2\sqrt{11} + 11)\cos\omega_2 t \qquad \text{(E.8)}$$

The solution obtained for 10 time steps using Eq. (E.8) are summarized in Table E.16.

TABLE E.16

t	x_1	x_2
0.308	0.114418	0.114879
0.616	0.410331	0.417369
0.924	0.764079	0.796983
1.232	1.023732	1.116667
1.54	1.068416	1.264493
1.848	0.856177	1.195575
2.156	0.441968	0.94829
2.464	−0.04126	0.628667
2.772	−0.4313	0.369003
3.08	−0.59686	0.276542

To use R-K method, use the following set of equations:

Here, $\dot{Y} = f(x_1, x_2, x_3, x_4, t)$, where $Y = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$ and $f(x_1, x_2, x_3, x_4, t) = \begin{Bmatrix} x_3 \\ x_4 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix}$

Where

$$\begin{Bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = -[M]^{-1}[K]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} - [M]^{-1}[C]\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix} + [M]^{-1}F(t)$$

In total, it can be written as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{Bmatrix} + \begin{bmatrix} [0] \\ [M]^{-1}F(t) \end{bmatrix}$$

or $$\dot{Y} = [E]Y + F$$

Problem E.17

Find the response of the two degrees of freedom system, when $F_1(t) = \sin 2t$ and $F_2(t) = \sin 2t$, using the fourth-order Runge Kutta method. The differential equations of this system are given as

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t$$

with the initial conditions $$x_0 = \begin{Bmatrix} 0 \\ 0.1 \end{Bmatrix}, \quad \dot{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{m/s.}$$

Solution:

The given equations can be written as

$$2\ddot{x}_1 + 3\dot{x}_1 - 0.5\dot{x}_2 + 3x_1 - x_2 = \sin 2t$$

$$\ddot{x}_2 - 0.5\dot{x}_1 + 0.5\dot{x}_2 - x_1 + x_2 = \sin 2t$$

Substituting

$$\dot{x}_1 = y_1 = f_1(x_1, x_2, y_1, y_2, t) \quad \text{and} \quad \dot{x}_2 = y_2 = f_2(x_1, x_2, y_1, y_2, t) \tag{1}$$

we get

$$2\dot{y}_1 + 3y_1 - 0.5y_2 + 3x_1 - x_2 = \sin 2t \tag{2}$$

or $$\dot{y}_1 = \phi_1(x_1, x_2, y_1, y_2, t)$$

and

$$\dot{y}_2 - 0.5y_1 + 0.5y_2 - x_1 + x_2 = \sin 2t \tag{3}$$

or $$\dot{y}_2 = \phi_2(x_1, x_2, y_1, y_2, t)$$

Solving these first-order equations using Runge-Kutta's method with the given initial conditions gives

$$x_{01} = 0, \quad x_{02} = 0.1$$

and

$$\dot{x}_{01} = 1 \quad \text{and} \quad \dot{x}_{02} = 0$$

We obtain the results for time t = 0 to 10 s as shown in the following Table E.17.

TABLE E.17

t	x_1	x_2	$\dot{x}_1$	$\dot{x}_2$
0	0	0.1	1	0
1	0.4550	0.2500	0.0790	0.4250
2	0.5470	0.9850	–0.0118	0.6490
3	0.3490	0.9050	–0.2770	0.6510
4	0.0705	0.0604	–0.1710	–0.7250
5	0.0888	–0.0464	0.03210	0.1990
6	–0.0183	–0.0985	–0.1950	–0.2790
7	–0.2330	–0.5300	–0.0880	–0.2770
8	–0.0785	–0.1860	0.2250	0.6550
9	0.0408	0.2270	–0.00143	0.0825
10	–0.0839	–0.1090	–0.0781	–0.4010

Problem E.18

Calculate the response of the system considered in Problem E.16 using the Houbolt method.

Solution:

Taking the time step $\Delta t = 0.308$, we then have the following

$$a_0 = \frac{2}{(\Delta t)^2} = 21.08$$

$$a_1 = \frac{11}{6\Delta t} = 5.952$$

$$a_2 = \frac{5}{(\Delta t)^2} = 52.707$$

$$a_3 = \frac{3}{\Delta t} = 9.74$$

$$a_4 = -2a_0 = -42.166$$

$$a_5 = -a_3/2 = -4.87$$

$$a_6 = \frac{a_0}{2} = 10.5415$$

$$a_7 = a_3/9 = 1.082$$

We need $X_{\Delta t}$ and $X_{2\Delta}$ to start the integration. We use the values calculated with the central difference method in Problem E.4, i.e.,

$$\{X_{\Delta t}\} = \begin{bmatrix} 0.1187 \\ 0.1187 \end{bmatrix}$$

$$\{X_{2\Delta t}\} = \begin{bmatrix} 0.424 \\ 0.429 \end{bmatrix}$$

Next, we compute $\left[\bar{K}\right]$ and obtain

$$\left[\bar{K}\right] = \left[K\right] + a_0\left[M\right] + a_1\left[C\right]$$

$$= \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} + 21.08\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + 5.952\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 46.166 & 5 \\ 2.0 & 48.166 \end{bmatrix}$$

$$\left[\bar{K}\right]^{-1} = \begin{bmatrix} 0.0218 & -2.226 \times 10^{-6} \\ -9.03 \times 10^{-4} & 0.0209 \end{bmatrix}$$

For each time step, we need $\left\{\bar{F}_{t+\Delta t}\right\}$ that is given by

$$\left\{\bar{F}_{t+\Delta t}\right\} = \left\{F_{t+\Delta t}\right\} + \left[M\right]\left(a_2\left\{X_t\right\} + a_4\left\{X_{t-\Delta t}\right\} + a_6\left\{X_{t-2\Delta t}\right\}\right)$$

$$+ \left[C\right]\left(a_3\left\{X_t\right\} + a_5\left\{X_{t-\Delta t}\right\} + a_7\left\{X_{t-2\Delta t}\right\}\right)$$

$$= \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\left(52.707\begin{bmatrix} 0.424 \\ 0.429 \end{bmatrix} - 42.166\begin{bmatrix} 0.1187 \\ 0.1187 \end{bmatrix} + 10.5415\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\left(9.74\begin{bmatrix} 0.424 \\ 0.429 \end{bmatrix} - 4.87\begin{bmatrix} 0.1187 \\ 0.1187 \end{bmatrix} + 1.082\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 39.686 \\ 40.212 \end{bmatrix}$$

We solve $\left\{X_{t+\Delta t}\right\}$ for each time step from the following equation

$$\left[\bar{K}\right]\left\{X_{t+\Delta t}\right\} = \left\{\bar{F}_{t+\Delta t}\right\}$$

or

$$\left\{X_{t+\Delta t}\right\} = \left[\bar{K}\right]^{-1}\left\{\bar{F}_{t+\Delta t}\right\}$$

The solution obtained for 10 time steps is summarized in Table E.18

TABLE E.18

Time	**Δt**	**$2\Delta t$**	**$3\Delta t$**	**$4\Delta t$**	**$5\Delta t$**
X_1	0.1186	0.4406	0.8670	1.300	1.664
X_2	0.1186	0.4125	0.7089	0.8500	0.7308
Time	**$6\Delta t$**	**$7\Delta t$**	**$8\Delta t$**	**$9\Delta t$**	**$10\Delta t$**
X_1	1.913	2.045	2.087	2.084	2.082
X_2	0.3266	−0.3011	−1.024	−1.683	−2.135

The method identifies the solution for the following

$$[M]\{\ddot{u}_{n+1}\} + \{f_{int_{n+1}}\} = \{f_{n+1}\}$$

$$\{\dot{u}_{n+1}\} = \frac{1}{6\Delta t}\left[11\{u_{n+1}\} - 18\{u_n\} + 9\{u_{n-1}\} - 2\{u_{n-2}\}\right]$$

$$\{\ddot{u}_{n+1}\} = \frac{1}{\Delta t^2}\left[2\{u_{n+1}\} - 5\{u_n\} + 4\{u_{n-1}\} - \{u_{n-2}\}\right]$$

Problem E.19

Calculate the displacement response of the system considered in problem E.12 using the Newmark method. Use $\alpha = 0.25$ and $\beta = 0.5$.

Solution:

Following the steps of calculations given in Table E.4, we have

$$\{X_0\} = \{\dot{X}_0\} = 0 \quad \text{and} \quad \{\ddot{X}_0\} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

The integration constants are with time step $\Delta t = 0.308$.

$$a_0 = \frac{1}{\alpha(\Delta t)^2} = \frac{1}{0.25(0.308)^2} = 42.17$$

$$a_1 = \frac{\beta}{\alpha\Delta t} = \frac{0.5}{0.25(0.308)} = 6.49$$

$$a_2 = \frac{1}{\alpha\Delta t} = \frac{1}{0.25(0.308)} = 12.99$$

$$a_3 = \left(\frac{1}{2\alpha} - 1\right) = \frac{1}{2(0.25)} - 1 = 1$$

$$a_4 = \frac{\beta}{\alpha} - 1 = \frac{0.5}{0.25} - 1 = 1$$

$$a_5 = \frac{\Delta t}{2}\left(\frac{\beta}{\alpha} - 2\right) = \frac{0.308}{2}\left(\frac{0.5}{0.25} - 2\right) = 0$$

$$a_6 = \Delta t(1 - \beta) = 0.308\,(1 - 0.5) = 0.154$$

$$a_7 = \beta\Delta t = 0.5\,(0.308) = 0.154$$

The effective stiffness matrix is

$$[\bar{K}] = [K] + a_0[M] + a_1[C]$$

$$= \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} + 42.17\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + 6.49\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 88.34 & 5 \\ 2.0 & 90.34 \end{bmatrix}$$

For each time step, we need to evaluate

$$\{\bar{F}_{t+\Delta t}\} = \{F_{t+\Delta t}\} + [M]\left(a_0\{X_t\} + a_2\{\dot{X}_t\} + a_3\{\ddot{X}_t\}\right)$$

$$+ [C]\left(a_1\{X_t\} + a_4\{\dot{X}_t\} + a_5\{\ddot{X}_t\}\right)$$

$$= \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\left(42.17\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 12.99\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}\right)$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\left(6.49\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Then, solve for displacements at each time step.

$$[\bar{K}]\{X_{t+\Delta t}\} = \{\bar{F}_{t+\Delta t}\}$$

and compute

$$\{\ddot{X}_{t+\Delta t}\} = a_0\left(\{X_{t+\Delta t}\} - \{X_t\}\right) - a_2\{\dot{X}_t\} - a_3\{\ddot{X}_t\}$$

$$= 42.17\left(\{X_{t+\Delta t}\} - \{X_t\}\right) - 12.99\{\dot{X}_t\} - 1\{\ddot{X}_t\}$$

$$\{\dot{X}_{t+\Delta t}\} = \{\dot{X}_t\} + 0.154\{\ddot{X}_t\} + 0.154\{\ddot{X}_{t+\Delta t}\}$$

The solution obtained is given in Table E.19 for 10 time steps.

TABLE E.19

Time	**Δt**	**$2\Delta t$**	**$3\Delta t$**	**$4\Delta t$**	**$5\Delta t$**
X_1	0.113	0.423	0.849	1.290	1.651
X_2	0.109	0.382	0.677	0.822	0.681

Time	**$6\Delta t$**	**$7\Delta t$**	**$8\Delta t$**	**$9\Delta t$**	**$10\Delta t$**
X_1	1.882	0.212	1.992	1.986	2.022
X_2	0.212	−0.509	−1.310	−1.979	−2.336

MATLAB Program:

```
K=[4 5;2 6];
M=[2 0;0 2];
C=[0.0 0.0;0.0 0.0];
dt=0.05;T=10;
X0=[0;0];X0d=[0;0];F=[5;5];
X0dd=inv(M)*(F-C*X0d-K*X0);
beta=0.5; gama=0.25; %0.25*(0.5+beta);
a0=1/(beta*dt^2); a1=gama/ (beta*dt); a2=1/(beta*dt);
a3= (1/2*beta)-1; a4= (gama/beta-1); a5=0.5*(gama/beta-2)*dt;
a6=dt*(1-beta);a7=beta*dt;
Kb=K+a0*M+a1*C;
```

```
i=1;
X(:,1)=X0;Xd(:,1)=X0d;Xdd(:,1)=X0dd;t=0;
fprintf('time(s)\t\tX1\t\tX2\n');
fprintf('%f\t%f\t%f\n',t,X(1,1),X(2,1));
for t=dt:dt:T
   i=i+1;
   F=[5;5];
   Fb=F+M*(a0*X(:,i-1)+a2*Xd(:,i-1)+a3*Xdd(:,i-1))+C*(a1*X(:,
     i-1)+a4*Xd(:,i-1)+a5*Xdd(:,i-1));
   X(:,i)=inv(Kb)*Fb;
   Xdd(:,i)=a0*(X(:,i)-X(:,i-1))-a2*Xd(:,i-1)-a3*Xdd(:,i-1);
   Xd(:,i)=a1*(X(:,i)-X(:,i-1))-a4*Xd(:,i-1)-a5*Xdd(:,i-1);
   fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
end
t=[0:dt:T];
plot(t,X(1,:),'-p',t,X(2,:),'-*')
xlabel('time(s)');
ylabel('displacement(m)');
legend('DOF-1','DOF-2');
grid on;
```

Problem E.20

Solve the following two degrees of freedom problem using Newmark Beta method.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{X} + \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume initial conditions as: $X_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \dot{X}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Solution:

Assuming $\Delta t = 0.25$ s, computing the acceleration vector:

$$\ddot{X}_0 = \begin{Bmatrix} -2 \\ -2 \end{Bmatrix}$$

Choose $\alpha = 0.25$, $\partial = 0.5$ the constants are

$$a_1 = \frac{1}{\alpha \cdot \Delta t^2} = 64,\ a_2 = \frac{\partial}{\alpha \cdot \Delta t} = 8,\ a_3 = \frac{1}{\alpha \cdot \Delta t} = 16,\ a_4 = \frac{1}{2\alpha} - 1$$

$$a_5 = \frac{\partial}{\alpha} - 1 = 1,\ a_6 = \frac{\Delta t}{2}\left(\frac{\partial}{\alpha} - 2\right) = 0,\ a_7 = \Delta t(1 - \partial) = 0.125,\ a_8 = \partial \cdot \Delta t = 0.125$$

$$\therefore [\hat{K}] = [K] + a_1[M] = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} + 64\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 66 & 2 \\ 2 & 69 \end{bmatrix}$$

$$\hat{F}_{t+\Delta t} = [M](64X_t + 16\dot{X}_t + \ddot{X}_t) = 64X_t + 16\dot{X}_t + \ddot{X}_t$$

Because [M] is identify matrix.

Solve $X_{t+\Delta t} = [\hat{K}]^{-1} \cdot \hat{F}_{t+\Delta t}$

$$\ddot{X}_{t+\Delta t} = 64(X_{t+\Delta t} - X_t) - 16\dot{X}_t - \ddot{X}_t$$

$$\dot{X}_{t+\Delta t} = \dot{X}_t + 0.125\ddot{X}_t + 0.125\ddot{X}_{t+\Delta t}$$

The values of $\{X\}$ computed for various values of t are shown in Table E.20.

TABLE E.20

	X	
Time = i · Δt	**x_1**	**x_2**
0	1	0
0.25	0.939	−0.056
0.5	0.7699	−0.202
0.75	0.529	−0.375
1.0	0.264	−0.497
1.25	0.0217	−0.504
1.5	0.136	−0.363
1.75	−0.243	−0.107
2.00	−0.373	0.242
2.25	−0.499	0.271

It is important to note that unless ∂ is taken as 1/2, there is a spurious damping introduced proportional to $(\partial - 1/2)$. If ∂ is taken as zero, a negative damping results; this involves a self-excited vibration arising due to numerical procedure. Similarly, if ∂ is greater then 1/2, a positive damping is introduced. This reduces the magnitude of response even without real damping in the problem. The method is unconditionally stable for $\alpha > \frac{1}{2}\left(\partial + \frac{1}{2}\right)^2$ and $\partial \geq 1/2$.

Problem E.21

The equations of motion of a two degrees of freedom system are given by

$$2\ddot{x}_1 + 6x_1 - 2x_2 = 5$$

$$\ddot{x}_2 - 2x_1 + 4x_2 = 20\sin 5t$$

Assuming the initial conditions are $\dot{x}_1(0) = x_1(0) = \dot{x}_2(0) = x_2(0) = 0$, find the response of the system using the central difference method with $\Delta t = 0.25$s.

Solution:

The system is represented with mass and stiffness matrices given below:

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}, \quad \text{with} \quad F(t) = \begin{Bmatrix} 5 \\ 20\sin 5t \end{Bmatrix}$$

The undamped natural frequencies and the mode shapes of the system can be found by solving the eigenvalue problem:

$$\left\{\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} - \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The solution of above equation is given by

$$\omega_1 = 1.414 \text{ and } \omega_2 = 2.236$$

Hence, the actual (natural) periods of the system are

$$\tau_1 = \frac{2\pi}{\omega_1} = 4.44 \quad \text{and} \quad \tau_2 = \frac{2\pi}{\omega_2} = 2.809$$

It requires to select time step (Δt) as $<= \tau_2/10 = 0.2809$,

The initial value of acceleration is

$$\ddot{X}_0 = [M]^{-1}\{F - [K]X_0\}$$

For every time-step, the value of X_{-1} as follows:

$$X_{-\Delta t} = X_0 - \Delta t\,\dot{X}_0 + \frac{\Delta t^2}{2}\ddot{X}_0$$

$$\hat{f}(t) = F(t) - \left(K - \frac{2}{\Delta t^2}M\right)X_t - \left(\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right)X_{t-\Delta t}$$

and $$X_{t+\Delta t} = \left[\frac{1}{\Delta t^2}[M] + \frac{1}{2\Delta t}[C]\right]^{-1}\hat{f}(t)$$

Now these equations can be applied recursively to obtain $X_1, X_2, \ldots$ The results obtained are shown in Table E.21 and Fig. E.21.

TABLE E.21

	X	
Time $t_i = i \cdot \Delta t$	x_1	x_2
0.000000	0.00000	0.00000
0.250000	0.07812	0.00000
0.500000	0.29785	1.19599
0.750000	0.69273	2.87831

MATLAB Program:

```
M=[2 0;0 1];
K=[6 -2;-2 4];
C=[0.0 0.0;0.0 0.0];
dt=0.25;
  x0=[0;0];x0d=[0;0];
```

```
  F0=[5;0];
  T=4;
  x0dd=inv(M)*(F0-C*x0d-K*x0);
  xprev=x0-(dt*x0d)+((dt^2)*(x0dd/2));
  a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
  mbar=(a0*M)+(a1*C);
t=0;
v(:,1)=x0d;a(:,1)=x0dd;
i=1;
fprintf('time\t\tX(1)\t\tX(2)\n');
for t=0:dt:T+dt
X(:,i)=x0;
F=[5;20*sin(5*t)];
Fbar=F+(a2*M-K)*x0+(a1*C-a0*M)*xprev;
x=inv(mbar)*Fbar;
xprev=x0;
x0=x;
fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
i=i+1;
p=i;
end
```

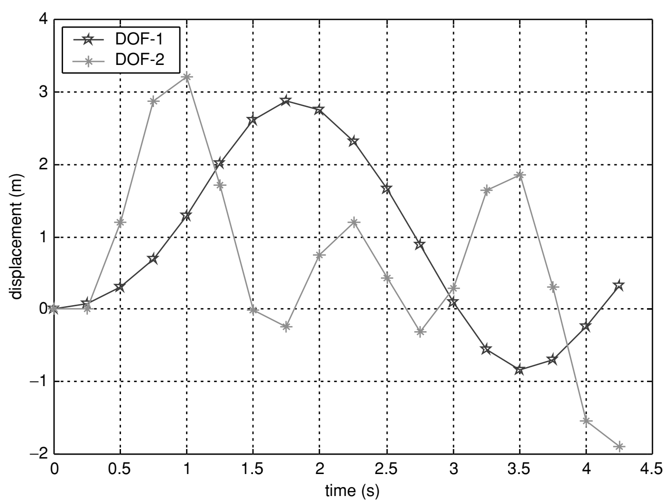


Figure E.21

Problem E.22

Find the response of a system shown in Fig. E.22, when the forcing functions are given by $F_1(t) = 0$ and $F_2(t) = 10$. Assume the zero initial conditions.

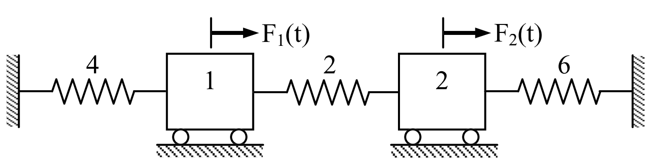


Figure E.22

Use Newmark method with $\alpha = 1/6$ and $\beta = 1/2$.

Solution:

The system is represented with mass and stiffness matrices given below:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}, \quad \text{with} \quad F(t) = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

The following steps are followed:

$$\left(\frac{1}{\alpha\Delta t^2}\left[M\right] + \left[K\right]\right)U_{n+1} = R_{n+1} + \left[M\right]\left(\frac{1}{\alpha\Delta t^2}U_n + \frac{1}{\alpha\Delta t}\dot{U}_n + \left(\frac{1}{2\alpha} - 1\right)\ddot{U}_n\right)$$

The initial conditions are: $U_0 = 0,\ \dot{U}_0 = 0,\ \ddot{U}_0 = \left[M\right]^{-1}\left(R_0 - \left[K\right]U_0\right)$

$$\ddot{U}_{n+1} = \frac{1}{\alpha\Delta t^2}\left[U_{n+1} - U_n\right] - \frac{1}{\alpha\Delta t}\dot{U}_n - \left(\frac{1}{2\alpha} - 1\right)\ddot{U}_n$$

$$\dot{U}_{n+1} = \beta\Delta t\,\ddot{U}_{n+1} + \left(1 - \beta\right)\Delta t\ddot{U}_n + \dot{U}_n$$

MATLAB Program:

```
K=[6 -2;-2 8];
M=[1 0;0 2];
C=[0.0 0.0;0.0 0.0];
dt=0.05;T=0.2;
X0=[0;0];X0d=[0;0];F=[0;10];
X0dd=inv(M)*(F-C*X0d-K*X0);
beta=0.5;gama=1/6;%0.25* (0.5+beta);
a0=1/(beta*dt^2); a1=gama/ (beta*dt); a2=1/(beta*dt);
a3= (1/2*beta)-1; a4= (gama/beta-1); a5=0.5*(gama/beta-2)*dt;
a6=dt*(1-beta);a7=beta*dt;
Kb=K+a0*M+a1*C;
i=1;
X(:,1)=X0;Xd(:,1)=X0d;Xdd(:,1)=X0dd;t=0;
fprintf('time(s)\t\tX1\t\tX2\n');
fprintf('%f\t%f\t%f\n',t,X(1,1),X(2,1));
for t=dt:dt:T
   i=i+1;
   F=[0;10];
   Fb=F+M*(a0*X(:,i-1)+a2*Xd(:,i-1)+a3*Xdd(:,i-1))+C*(a1*X
      (:,i-1)+a4*Xd(:,i-1)+a5*Xdd(:,i-1));
   X(:,i)=inv(Kb)*Fb;
   Xdd(:,i)=a0*(X(:,i)-X(:,i-1))-a2*Xd(:,i-1)-a3*Xdd(:,i-1);
   Xd(:,i)=a1*(X(:,i)-X(:,i-1))-a4*Xd(:,i-1)-a5*Xdd(:,i-1);
   fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
end
```

TABLE E.22

	X	
Time $t_i = i \cdot \Delta t$	**x_1**	**x_2**
0.000000	0.00000	0.00000
0.050000	0.00000	0.00155
0.100000	0.00003	0.01398
0.150000	0.00012	0.03725

The results obtained are shown in Fig. E.22(a).

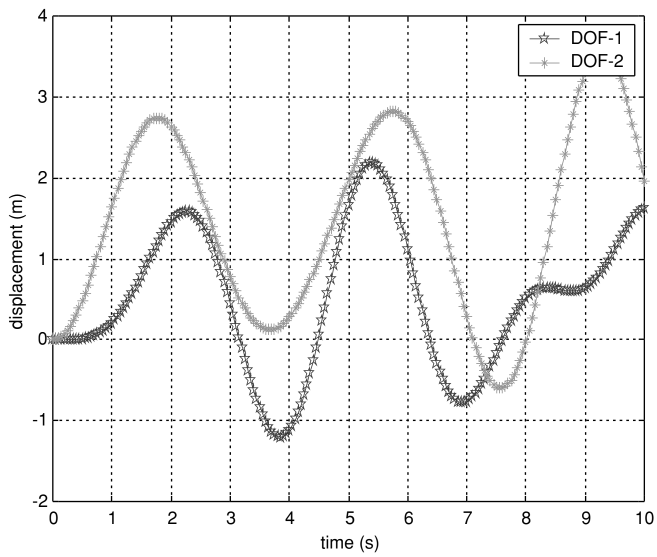


Figure E.22 (a)

Problem E.23

Obtain the response of the system given in Problem E.22 using the Wilson-theta method with $\theta = 1.4$.

Solution:

First, solve

$$\left(\frac{6}{(\theta\Delta t)^2}\left[M\right]+\left[K\right]\right)U_{n+\theta} = R_n + \theta\left(R_{n+1} - R_n\right) + \left[M\right]\left(\frac{6}{(\theta\Delta t)^2}U_n + \frac{6}{\theta\Delta t}\dot{U}_n + 2\ddot{U}_n\right)$$

Then we get

$$\ddot{U}_{n+1} = \frac{6}{\theta^3\Delta t^2}\left[U_{n+\theta} - U_n\right] - \frac{6}{\theta\Delta t}\dot{U}_n + \left(1 - \frac{3}{\theta}\right)\ddot{U}_n,$$

and

$$\dot{U}_{n+1} = \frac{\Delta t}{2}\left[\ddot{U}_{n+1} - \ddot{U}_n\right] + \dot{U}_n$$

The last step is to get

$$U_{n+1} = \frac{\Delta t^2}{6}\left[\ddot{U}_{n+1} + 2\ddot{U}_n\right] + \Delta t\dot{U}_n + U_n$$

The initial conditions are:

$$U_0 = 0, \quad \dot{U}_0 = 0, \quad \ddot{U}_0 = \left[M\right]^{-1}\left(R_0 - \left[K\right]U_0\right).$$

```
K=[6 -2;-2 8];
M=[1 0;0 2];
C=[0 0;0 0];
dt=0.05;T=10;
X0=[0;0];X0d=[0;0];F0=[0;10];
X0dd=inv(M)*(F0-C*X0d-K*X0);
theta=1.4;
a0=6/(theta*dt)^2;a1=3/(theta*dt);a2=2*a1;
a3=2;a4=(1/2*theta*dt);a5=-a2/theta;
a6=1-3/theta;
a7=dt/2;a8=dt^2/6;
Kb=K+a0*M+a1*C;
i=1;
X(:,1)=X0;Xd(:,1)=X0d;Xdd(:,1)=X0dd;t=0;
fprintf('time(s)\t\tX1\t\tX2\n');
fprintf('%f\t%f\t%f\n',t,X(1,1),X(2,1));
for t=dt:dt:T
   i=i+1;
   F=[0;10];
   Ftb=F0+M*(a0*X(:,i-1)+a2*Xd(:,i-1)+a3*Xdd(:,i-1))+C*(a1*X(:,i-1)
      +a3*Xd(:,i-1)+a4*Xdd(:,i-1))+theta*(F-F0);
   Xt(:,i)=inv(Kb)*Ftb;
   Xdd(:,i)=(a0/theta)*(Xt(:,i)-X(:,i-1))+a5*Xd(:,i-1)+a6*Xdd(:,i-1);
   Xd(:,i)=Xd(:,i-1)+a7*(Xdd(:,i)+Xdd(:,i-1));
   X(:,i)=X(:,i-1)+dt*Xd(:,i-1)+a8*(Xdd(:,i)+2*Xdd(:,i-1));
   F0=F;
   fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
end
t=[0:dt:T];
plot(t,X(1,:),'-p',t,X(2,:),'-*')
xlabel('time(s)');
ylabel('displacement(m)');
legend('DOF-1','DOF-2');
grid on;
```

The results obtained are shown in Fig. E.23.

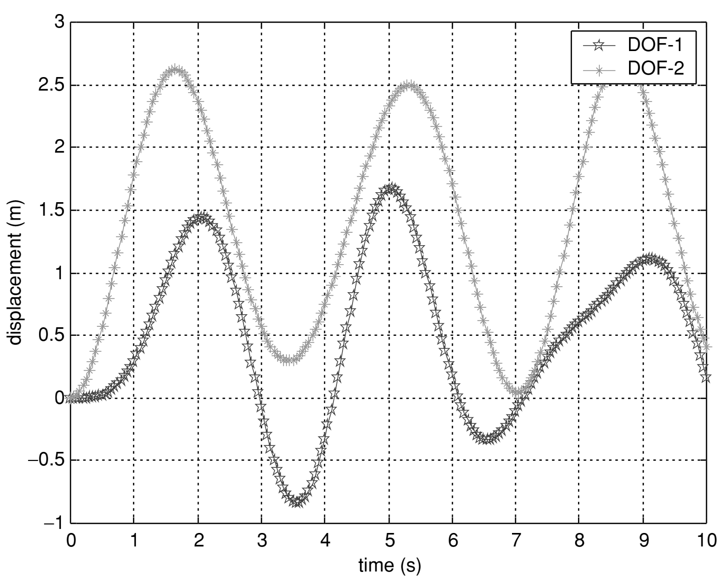


Figure E.23

TABLE E.23

Time $t_i = i \cdot \Delta t$	X x_1	X x_2
0.000000	0.00000	0.00000
0.050000	0.00000	0.00623
0.100000	0.00006	0.02486
0.150000	0.00027	0.05569

Problem E.24

Find the free vibration response of two degrees of freedom system shown in Fig. E.24 using Houbolt method. Assume $X(0) = \{0.2, 0\}^T$, $\dot{X}(0) = \{0, 0\}^T$.

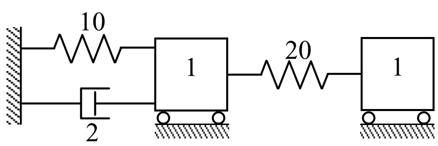


Figure E.24

Solution:

Writing equations of motion, the following matrices are obtained:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 30 & -20 \\ -20 & 20 \end{bmatrix}, \quad \text{with} \quad F(t) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$X(0) = \begin{Bmatrix} 0.2 \\ 0 \end{Bmatrix} \quad \text{and} \quad \dot{X}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

MATLAB Program:

```
K=[30 -20;-20 20];
M=[1 0;0 1];
C=[2 0;0 0];
dt=0.05;T=2;
X0=[0.2;0];X0d=[0;0];F=[0;0];
t=[0:dt:T];
X(:,2)=X0;
X0dd=inv(M)*(F-C*X0d-K*X0);
% USING CENTRAL DIFFERENCE METHOD TO OBTAIN PREVIOUS 3 VALUES
Xprev=X0-(dt*X0d)+((dt^2)*(X0dd/2));
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=(a0*M)+(a1*C);
kbar=(K-a2*M);
cbar=(a0*M-a1*C);
X(:,1)=X0;
Fbar=F-kbar*X0-cbar*Xprev;
X(:,2)=inv(mbar)*Fbar;
Fbar=F-kbar*X(:,2)-cbar*X0;
X(:,3)=inv(mbar)*Fbar;
% HOUBOLT METHOD BEGINS
a0=2/(dt^2);a1=11/(6*dt);a2=5/(dt^2);a3=3/dt;a4=-2*a0;
```

```
a5=-a3/2;a6=a0/2;a7=a3/9;
Kb=K+a0*M+a1*C;
p=3;
for i=3:length(t)
   F=[0;0];% F(t+2dt)
   Fb=F+M*(a2*X(:,i)+a4*X(:,i-1)+a6*X(:,i-2))+ C*(a3*X(:,i)+
      a5*X(:,i-1)+a7*X(:,i-2));
   X(:,i+1)=inv(Kb)*Fb;
   Xdd(:,i+1)=a0*X(:,i+1)-a2*X(:,i)-a4*X(:,i-1)-a6*X(:,i-2);
   Xd(:,i+1)=a1*X(:,i+1)-a3*X(:,i)-a5*X(:,i-1)-a7*X(:,i-2);
   p=p+1;
end
fprintf('\ntime\t\tX1\t\tX2\n');
for i=1:p
   time(i)=(i-1)*dt;
   fprintf('%f\t%f\t%f\n',time(i),X(1,i),X(2,i))
end
plot(time,X(1,:),'-p',time,X(2,:),'-*');
grid on;
xlabel('time(s)');
ylabel('displacement (m)');
legend('DOF-1','DOF-2');
```

The results obtained are shown in Fig. E.24(a).

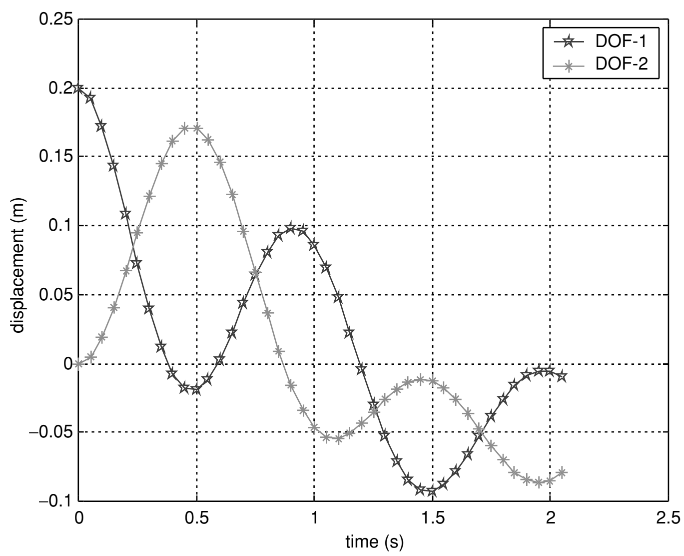


Figure E.24 (a)

TABLE E.24

Time $t_i = i \cdot \Delta t$	X x_1	x_2
0.000000	0.20000	0.00000
0.050000	0.19250	0.00500
0.100000	0.17220	0.01937
0.150000	0.14281	0.04098

Additional Problems and Solutions:

Problem AE.1

Find the response of a viscously damped single degree of freedom system subjected to a force

$$F(t) = F_0\left(1 - \sin\frac{\pi t}{2t_0}\right)$$

with the following data: $F_0 = 2N$, $t_0 = \pi$ s, m = 2 kg, c = 0.3 Ns/M, and k = 1 N/m. The values of the displacement and velocity of the mass at t = 0 are zero. Use the central difference method. Choose $\Delta t = 1$, 0.1 and 0.5 s and compares the results.

Solution:

This is a single degree of freedom system problem with all initial conditions zero.

% MATLAB Program:

```
% INITIAL VALUES
m=2;k=1;c=0.3;dt=0.1;
x0=0;x0d=0;
F0=2;
T=5;
x0dd=inv(m)*(F0-c*x0d-k*x0);
xprev=x0-(dt*x0d)+((dt^2)*x0dd/2);
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=a0*m+a1*c;
   t=0;
   v(1)=x0d;a(1)=x0dd;
   i=1;
   for t=0:dt:T+dt
   X(i)=x0;
   f=F0*(1-sin(0.5*t));
   fbar=f+(a2*m-k)*x0+(a1*c-a0*m)*xprev;
   x=inv(mbar)*fbar;
   xprev=x0;
   x0=x;
   i=i+1;
   p=i;
   end
   for i=2:p-1
      if i<p-1
      v(i)=(X(i+1)-X(i-1))/(2*dt);
      a(i)=(X(i+1)-2*X(i)+X(i-1))/dt^2;
      end
   end
   fprintf('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
   i=1;
   for t=0:dt:T
      fprintf('%f\t%f\t%f\t%f\n',t,X(i),v(i),a(i));
      i=i+1;
   end
   t=[0:dt:T+dt];
   plot(t,X,'-p');
      xlabel('Time(s)');
```

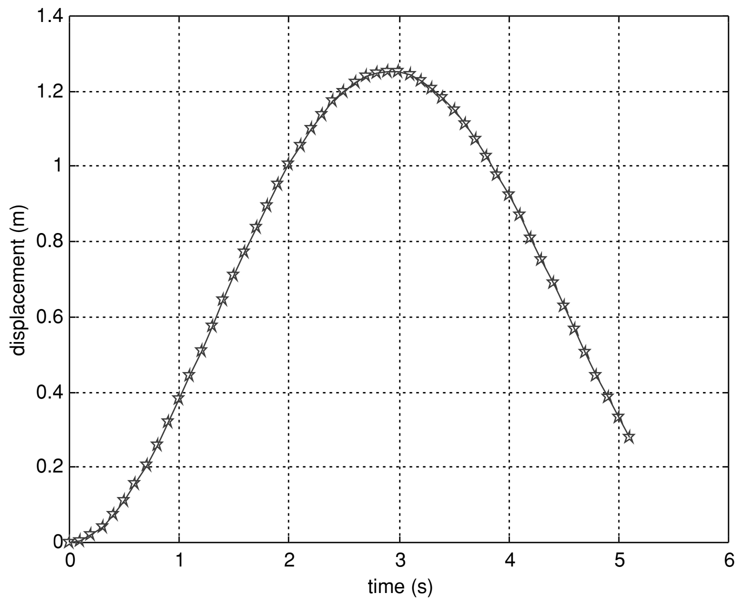


Figure AE.1 Displacement response plot for $\Delta t = 0.1$ s.

Problem AE.2

Find the solution of the equation $5\ddot{X} + 2.5\dot{X} + 4000X = F(t)$, where F(t) is as shown in Fig. AE.2 for the duration $0 \le t \le 1$. Assume that $X_0 = \dot{X}_0 = 0$ and $\Delta t = 0.05$. Use the central difference method.

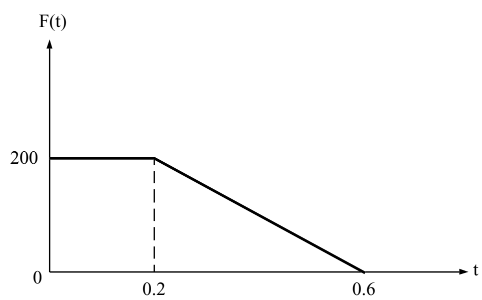


Figure AE.2

Solution:

% MATLAB Program:

```
% INITIAL VALUES
  m=5;k=4000;c=2.5;dt=0.05;
  x0=0;x0d=0;
  F0=200;
  T=1;
  x0dd=inv(m)*(F0-c*x0d-k*x0);
  xprev=x0-(dt*x0d)+((dt^2)*x0dd/2);
  a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
  mbar=a0*m+a1*c;
  t=0;
  v(1)=x0d;a(1)=x0dd;
  i=1;
  for t=0:dt:T+dt
```

```
X(i)=x0;
if t<=0.2 f=F0;
else if (t>0.2 & t<=0.6) f=-(F0/0.4)*(t-0.6);
   else if t>0.6 f=0;
      end
   end
end
fbar=f+(a2*m-k)*x0+(a1*c-a0*m)*xprev;
x=inv(mbar)*fbar;
xprev=x0;
x0=x;
i=i+1;
p=i;
end
for i=2:p-1
   if i<p-1
   v(i)=(X(i+1)-X(i-1))/(2*dt);
   a(i)=(X(i+1)-2*X(i)+X(i-1))/dt^2;
   end
end
fprintf('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
i=1;
for t=0:dt:T
   fprintf('%f\t%f\t%f\t%f\n',t,X(i),v(i),a(i));
   i=i+1;
end
t=[0:dt:T+dt];
plot(t,X,'-p');
xlabel('Time(s)');
ylabel('displacement(m)');
grid on;
```

Fig. AE.2(a) shows the response history.

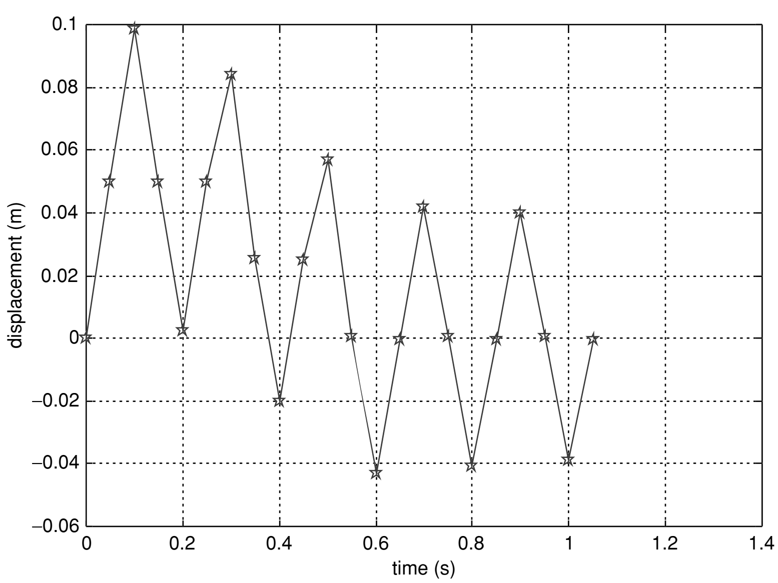


Figure AE.2 (a) MATLAB output for $\Delta t = 0.05$ s.

Problem AE.3

Solve numerically the differential equation $4\ddot{X} + 2000X = F(t)$ with the initial conditions $X_0 = \dot{X}_0 = 0$ and forcing function F(t) as shown in Fig. AE.3. Use central difference method, with $\Delta t = 0.02$ s.

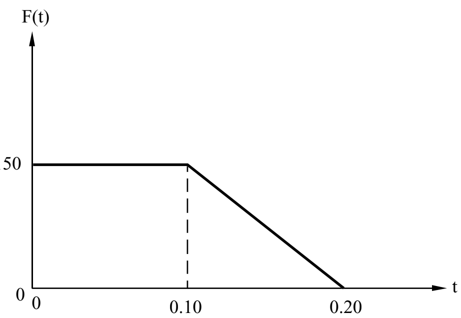


Figure AE.3

Solution:

% MATLAB Program:

```
% INITIAL VALUES
   m=4;k=2000;c=0;dt=0.02;
   x0=0;x0d=0;
   F0=150;
   T=1;
   x0dd=inv(m)*(F0-c*x0d-k*x0);
   xprev=x0-(dt*x0d)+((dt^2)*x0dd/2);
   a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
   mbar=a0*m+a1*c;
   t=0;
   v(1)=x0d;a(1)=x0dd;
   i=1;
   for t=0:dt:T+dt
   X(i)=x0;
   if t<=0.1 f=F0;
   else if (t>0.1 & t<=0.2) f=-(F0/0.1)*(t-0.2);
      else if t>0.2 f=0;
         end
      end
   end
   fbar=f+(a2*m-k)*x0+(a1*c-a0*m)*xprev;
   x=inv(mbar)*fbar;
   xprev=x0;
   x0=x;
   i=i+1;
   p=i;
   end
   for i=2:p-1
      if i<p-1
      v(i)=(X(i+1)-X(i-1))/(2*dt);
      a(i)=(X(i+1)-2*X(i)+X(i-1))/dt^2;
      end
   end
```

```
fprintf('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
i=1;
for t=0:dt:T
   fprintf('%f\t%f\t%f\t%f\n',t,X(i),v(i),a(i));
   i=i+1;
end
t=[0:dt:T+dt];
plot(t,X,'-p');
xlabel('Time(s)');
ylabel('displacement(m)');
grid on;
```

Fig. AE.3 (a) shows the response history.

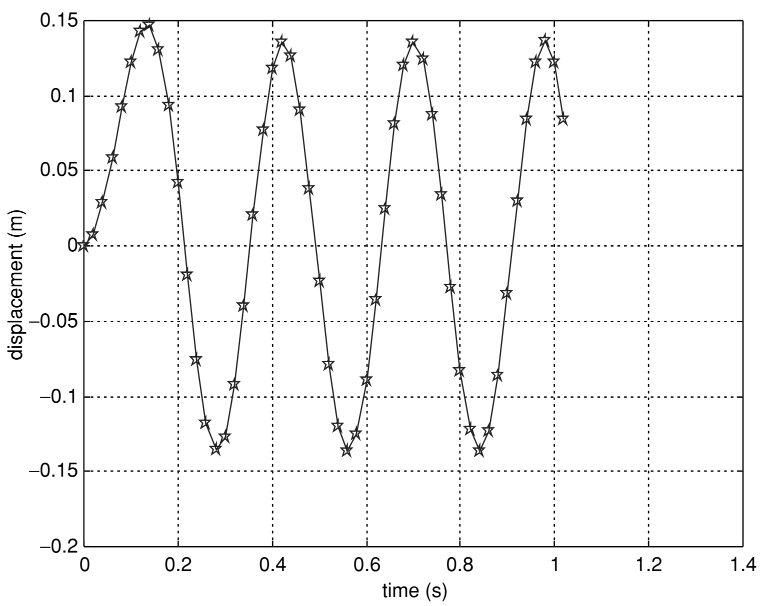


Figure AE.3 (a) Response history

Problem AE.4

Solve numerically the solution to the problem of a spring mass system excited by a triangular impulse. The differential equation of motion and the initial conditions are given as

$$0.5\ddot{X} + 8\pi^2 X = F(t)$$

with $$X_1 = \dot{X}_1 = 0$$

The triangular force is defined in Fig. AE.4. Use central difference method, with $\Delta t = 0.05$ s.

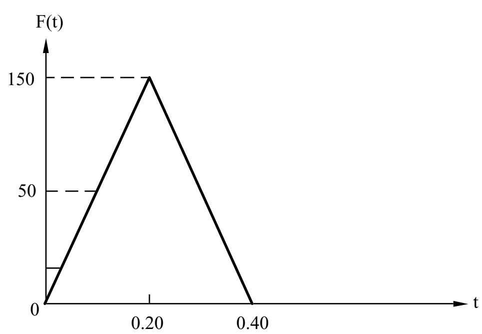


Figure AE.4

Solution:

% MATLAB Program:

```
% INITIAL VALUES
   m=0.5;k=8*pi^2;c=0;dt=0.05;
   x0=0;x0d=0;
   F0=0;F=150;
   T=1;
   x0dd=inv(m)*(F0-c*x0d-k*x0);
   xprev=x0-(dt*x0d)+((dt^2)*x0dd/2);
   a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
   mbar=a0*m+a1*c;
   t=0;
   v(1)=x0d;a(1)=x0dd;
   i=1;
   for t=0:dt:T+dt
   X(i)=x0;
   if t<=0.2 f=(F*t/0.2);
   else if (t>0.2 & t<=0.4) f=-(F/0.2).*(t-0.4);
   else if t>0.4 f=0;
         end
      end
   end
   fbar=f+(a2*m-k)*x0+(a1*c-a0*m)*xprev;
   x=inv(mbar)*fbar;
   xprev=x0;
   x0=x;
   i=i+1;
   p=i;
   end
   for i=2:p-1
      if i<p-1
      v(i)=(X(i+1)-X(i-1))/(2*dt);
      a(i)=(X(i+1)-2*X(i)+X(i-1))/dt^2;
      end
   end
   fprintf('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
   i=1;
   for t=0:dt:T
      fprintf('%f\t%f\t%f\t%f\n',t,X(i),v(i),a(i));
      i=i+1;
   end
   t=[0:dt:T+dt];
   plot(t,X,'-p');
   xlabel('Time(s)');
   ylabel('displacement(m)');
   grid on;
```

Fig. AE.4 shows the plot of the response versus time.

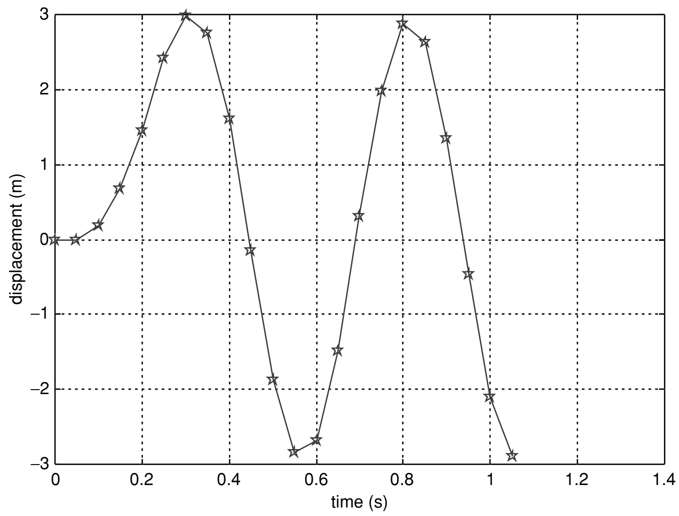


Figure AE.4 (a) MATLAB output for $\Delta t = 0.05$ s.

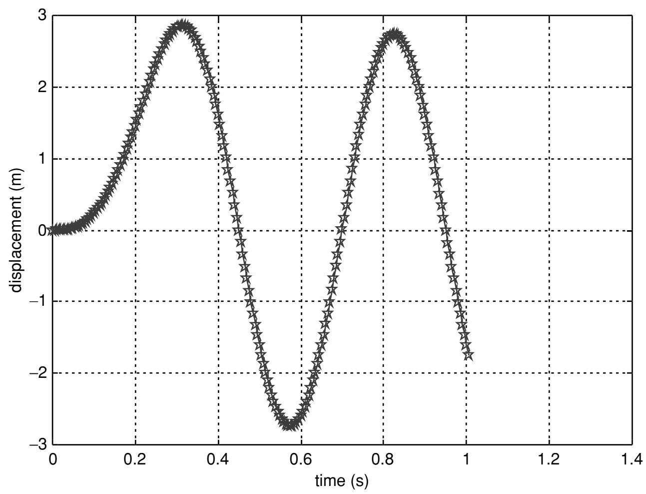


Figure AE.4 (b) MATLAB output for $\Delta t = 0.005$ s.

PROBLEM AE.5

Solve the following nonlinear vibration problem using the central difference method.

$$M\ddot{X} + C\dot{X} + KX + K^{*}X^{3} = F\cos\omega t$$

with $M = 1.0$, $C = 0.5$, $K = 1.0$, $K^{*} = 0.5$, $\Delta t = 0.05$, $t_{max} = 5.0$, and the initial conditions $X_0 = \dot{X}_0 = 0$. Plot the variation of X with t. Take $\omega = 1$ and $F = 10$.

Solution:

Here, the in X_{i+1} an additional term with $-K^{*}X_i^3$ will come and other things will remain same. Assuming F = 10N.

% MATLAB Program:

```
% INITIAL VALUES
   m=1;k=1;c=0.5;ks=0.5;dt=0.05;
   x0=0;x0d=0;omega=1;
   F0=10;
   T=5;
```

```
x0dd=inv(m)*(F0-c*x0d-k*x0);
xprev=x0-(dt*x0d)+((dt^2)*x0dd/2);
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=a0*m+a1*c;
t=0;
v(1)=x0d;a(1)=x0dd;
i=1;
for t=0:dt:T+dt
X(i)=x0;
f=F0*cos (omega*t);
% NON-LINEAR TERM
fbar=f+(a2*m-k)*x0+(a1*c-a0*m)*xprev-ks*(x0^3);
x=inv(mbar)*fbar;
xprev=x0;
x0=x;
i=i+1;
p=i;
end
for i=2:p-1
   if i<p-1
   v(i)=(X(i+1)-X(i-1))/(2*dt);
   a(i)=(X(i+1)-2*X(i)+X(i-1))/dt^2;
   end
end
fprintf('\ntime\t\tdisplacement\tvelocity\tacceleration\n');
i=1;
for t=0:dt:T
   fprintf('%f\t%f\t%f\t%f\n',t,X(i),v(i),a(i));
   i=i+1;
end
t=[0:dt:T+dt];
plot(t,X,'-p');
   xlabel('Time(s)');
```

Fig. AE.5 shows the MATLAB response.

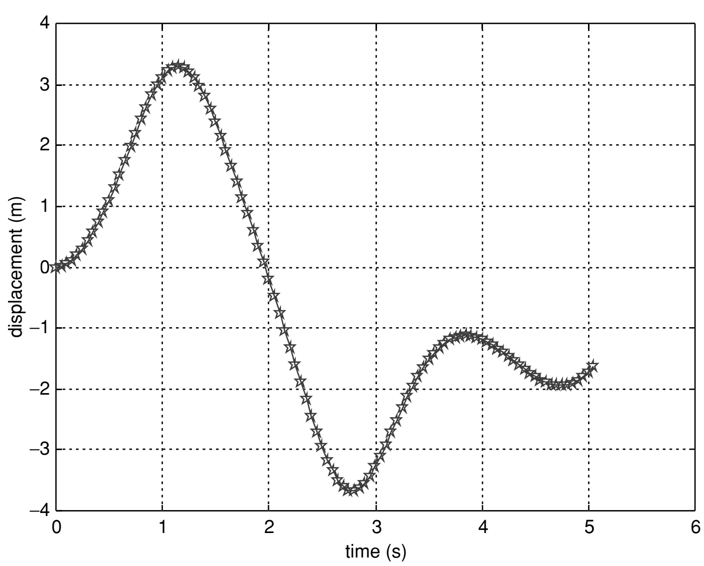


Figure AE.5 MATLAB output for $\Delta t = 0.05$ s.

PROBLEM AE.6

Solve Problem AE.1 using the fourthorder Runge-Kutta method.

Solution:

Here, $\dot{Y} = f(x_1,x_2,t)$ is a vector of functions

For single degree of freedom system, it contains

$$\dot{Y} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = f(x_1,x_2,t) = \begin{Bmatrix} x_2 \\ \dfrac{1}{m}\left(F(t) - kx_1 - cx_2\right) \end{Bmatrix}$$

Final solution takes the form

$$Y_{i+1} = Y_i + \frac{\Delta t}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right], \text{ where}$$

$$K_1 = f(x_1, x_2, t) = \begin{Bmatrix} p \\ q \end{Bmatrix}$$

$$K_2 = f(x_1 + p/2, x_2 + q/2, t + \Delta t/2) = \begin{Bmatrix} r \\ s \end{Bmatrix}$$

$$K_3 = f(x_1 + r/2, x_2 + q/2, t + \Delta t/2) = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$K_4 = f(x_1 + u, x_2 + v, t + \Delta t) = \begin{Bmatrix} m \\ n \end{Bmatrix}$$

%MATLAB Program:

```
dt=0.5;T=10;
h=dt;
x1=0;
x2=0;
i=1;
for t=0:h:T
f1=h*f(t,x1,x2); g1=h*g(t,x1,x2);
f2=h*f((t+h/2),(x1+f1/2),(x2+g1/2));g2=h*g((t+h/2),(x1+f1/2),
    (x2+g1/2));
f3=h*f((t+h/2),(x1+f2/2),(x2+g2/2));g3=h*g((t+h/2),(x1+f2/2),
    (x2+g2/2));
f4=h*f((t+h),(x1+f3),(x2+g3)); g4=h*g((t+h),(x1+f3),(x2+g3));
   x1=x1+((f1+f4)+2*(f2+f3))/6.0;
x2=x2+((g1+g4)+2*(g2+g3))/6.0;
X(i)=x1;
Y(i)=x2;
i=i+1;
end
t=[0:h:T];
plot(t,X,'-p',t,Y,'-*');
grid on;
```

```
xlabel('time(s)');
legend('displacement(m)','velocity(m/s)',2)
```

This program is executed with two other separate programs f·m and g·m given below:

```
% file f.m
function v1=f(t,x1,x2)
v1=x2;
% file g.m
function v2=g(t,x1,x2)
k=1; m=1; c=0;
F=100*(1-cos(t));
v2=(F-k*x1-c*x2)/m;
```

The output of the program is shown in Fig. AE.6.

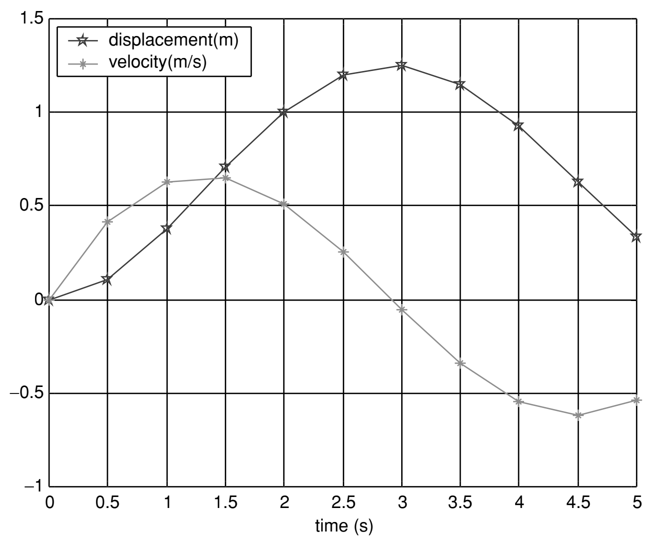


Figure AE.6 MATLAB output

PROBLEM AE.7

Solve Problem AE.2 using the fourth-order Runge-Kutta method.

Solution:

% MATLAB Program:

```
dt=0.05;T=1;
h=dt;
x1=0;        % displacement
x2=0;        % velocity
i=1;
fprintf('time\t\tdisplacement\tvelocity\n');
for t=0:h:T
fprintf('%f\t%f\t%f\n',t,x1,x2);
X(i)=x1;
Y(i)=x2;
f1=h*f(t,x1,x2); g1=h*g(t,x1,x2);
f2=h*f((t+h/2),(x1+f1/2),(x2+g1/2));g2=h*g((t+h/2),(x1+f1/2),
   (x2+g1/2));
f3=h*f((t+h/2),(x1+f2/2),(x2+g2/2));g3=h*g((t+h/2),(x1+f2/2),
   (x2+g2/2));
```

```
f4=h*f((t+h),(x1+f3),(x2+g3)); g4=h*g((t+h),(x1+f3),(x2+g3));
x1=x1+((f1+f4)+2*(f2+f3))/6.0;
x2=x2+((g1+g4)+2*(g2+g3))/6.0;
i=i+1;
end
time=[0:h:T];
plot(time,X,'-p');
grid on;
xlabel('time(s)');
ylabel('displacement(m)')
```

function g·m is given below:

```
function v2=g(t,x1,x2)
    k=4000; m=5; c=2.5;
  if t<=0.2 F=200;
  else if (t>0.2 & t<=0.6) F=-(200/0.4)*(t-0.6);
  else if t>0.6 F=0;
  end
  end
  end
  v2=(F-k*x1-c*x2)/m;
```

The displacement response is shown in Fig. AE.7

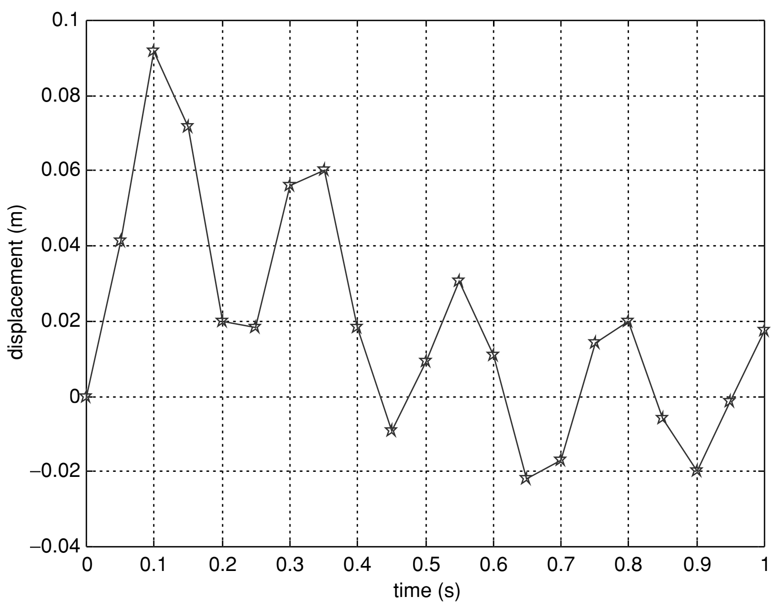


Figure AE.7 MATLAB output

Problem AE.8

Solve Problem AE.3 using the Runge-Kutta method.

Solution:

% MATLAB Program:

```
dt=0.02;T=1;
h=dt;
x1=0;        % displacement
x2=0;        % velocity
```

```
i=1;
fprintf('time\t\tdisplacement\tvelocity\n');
for t=0:h:T
fprintf('%f\t%f\t%f\n',t,x1,x2);
X(i)=x1;
Y(i)=x2;
f1=h*f(t,x1,x2); g1=h*g(t,x1,x2);
f2=h*f((t+h/2),(x1+f1/2),(x2+g1/2));g2=h*g((t+h/2),(x1+f1/2),
  (x2+g1/2));
f3=h*f((t+h/2),(x1+f2/2),(x2+g2/2));g3=h*g((t+h/2),(x1+f2/2),
  (x2+g2/2));
f4=h*f((t+h),(x1+f3),(x2+g3)); g4=h*g((t+h),(x1+f3),(x2+g3));
x1=x1+((f1+f4)+2*(f2+f3))/6.0;
x2=x2+((g1+g4)+2*(g2+g3))/6.0;
i=i+1;
end
time=[0:h:T];
plot(time,X,'-p');
grid on;
xlabel('time(s)');
ylabel('displacement(m)')
```

The function g·m defining the force signal is given below:

```
function v2=g(t,x1,x2)
  k=2000; m=4; c=0;
  if t<=0.1 F=150;
  else if (t>0.1 & t<=0.2) F=-(150/0.1)*(t-0.2);
  else if t>0.2 F=0;
  end
  end
  end
v2=(F-k*x1-c*x2)/m;
```

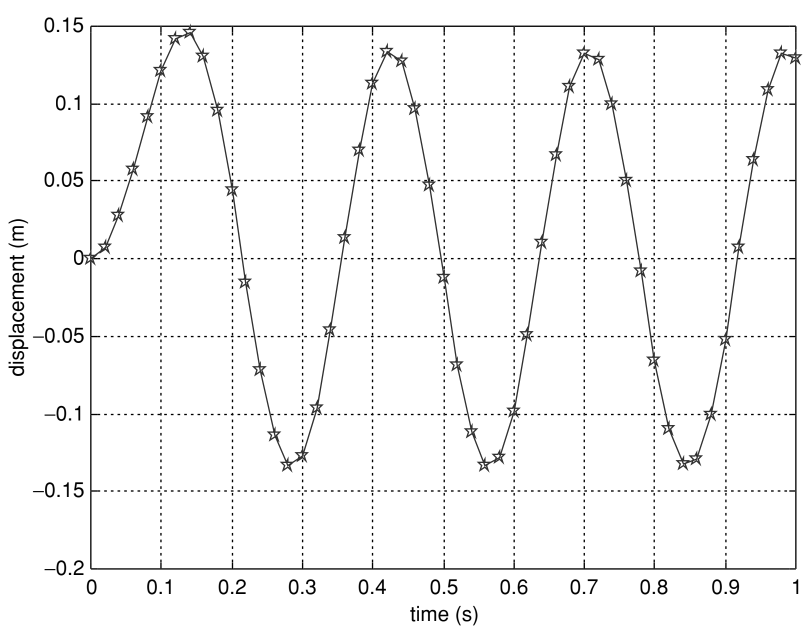


Figure AE.8 MATLAB output

Problem AE.9

Solve Problem AE.4 using the Runga-Kutta method.

Solution:

% MATLAB Program:

```
function v2=g(t,x1,x2)
  k=8*pi^2;m=0.5; c=0;
  if t<=0.2 F=(150*t/0.2);
  else if (t>0.2 & t<=0.4) F=-(150/0.2)*(t-0.4);
  else if t>0.4 F=0;
  end
  end
  end
  v2=(F-k*x1-c*x2)/m;
Here dt=0.05s and T=1 s.
```

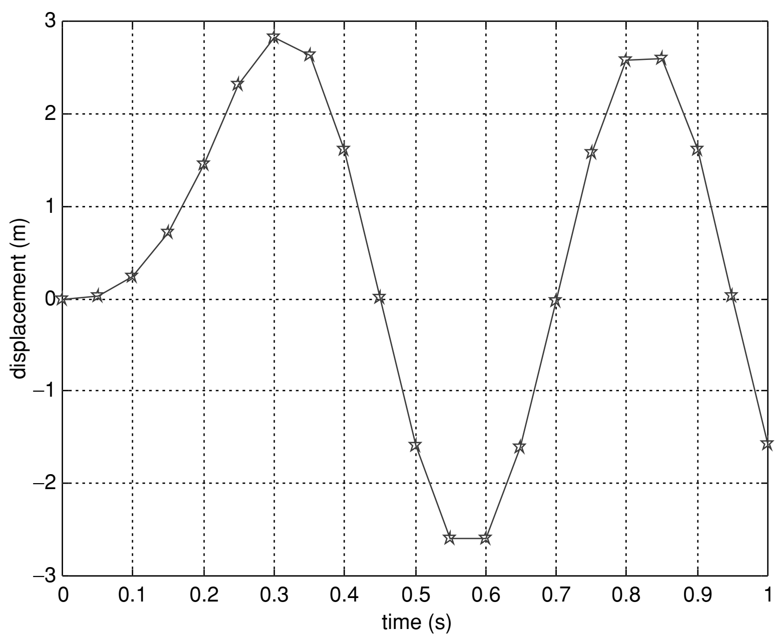


Figure AE.9 MATLAB output

Problem AE.10

Solve Problem AE.5 using the Runga-Kutta method.

Solution:

% MATLAB Program:

Here, the function defining the system g.m is given below:

```
% g.m
  function v2=g(t,x1,x2)
  k=1;m=1; c=0.5;omega=1;
  ks=0.5 % CUBIC STIFFNESS
  F=10*cos (omega*t);
  v2= (F-k*x1-c*x2-ks*x1^3)/m;
```

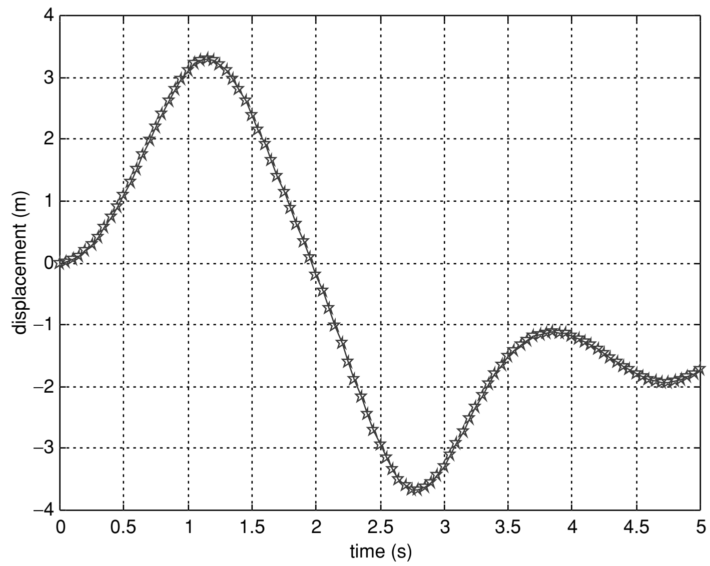


Figure AE.10 MATLAB output

Problem AE.11

Find the response of the two degrees of freedom system when $F_1(t) = 0$ and $F_2(t) = 10$ using the central difference method. The mass, stiffness, and damping matrices for this system are given as

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad [K] = \begin{bmatrix} 21 & -1 \\ -1 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 0.5 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}$$

All the initial conditions are given as zero. Use $\Delta t = 0.05$

Solution:

% MATLAB Program:

```
% INITIAL VALUES
  M=[1 0;0 10];
  K=[21 -1;-1 1];
  C=[0.5 -0.1;-0.1 0.1];
  dt=0.05;
  x0=[0;0];x0d=[0;0];
  F0=[0;10];
  T=2;
  x0dd=inv(M)*(F0-C*x0d-K*x0);
  xprev=x0-(dt.*x0d)+((dt^2).*(x0dd/2));
  a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
  mbar=(a0.*M)+(a1.*C);
  t=0;
  v(:,1)=x0d;a(:,1)=x0dd;
  i=1;
  fprintf('time\t\tX(1)\t\tX(2)\n');
  for t=0:dt:T+dt
  X(:,i)=x0;
```

```
F=F0;
Fbar=F+(a2.*M-K)*x0+(a1.*C-a0.*M)*xprev;
x=inv(mbar)*Fbar;
xprev=x0;
x0=x;
fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
i=i+1;
p=i;
end
for i=2:p-1
if i<p-1
v(:,i)=(X(:,i+1)-X(:,i-1)).*(1/(2*dt));
a(:,i)=(X(:,i+1)-2*X(:,i)+X(:,i-1)).*(1/dt^2);
end
end
t=[0:dt:T+dt];
plot(t,X(1,:),'-p',t,X(2,:),'-*');
xlabel('time(s)');
ylabel('displacement(m)');
legend('DOF-1','DOF-2',2);
grid on;
```

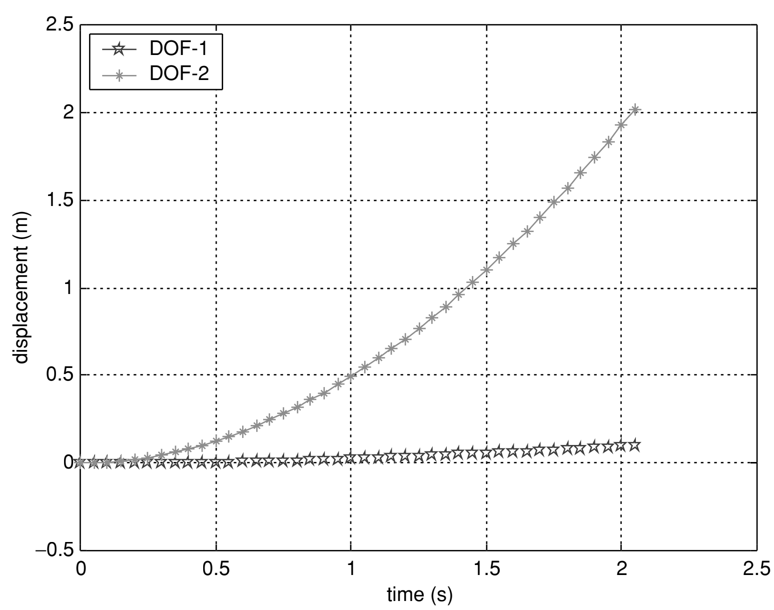


Figure AE.11 MATLAB output response

Problem AE.12

Solve Problem AE.11 using the two-cycle iteration with trapezoidal rule.

Solution:

For an undamped system, the following equations are applicable.

We now have:

$$\left(\frac{4}{\Delta t^2}\right)[M]+[K]U_{n+1}=R_{n+1}+[M]\left(\frac{4}{\Delta t^2}U_n+\frac{4}{\Delta t}\dot{U}_n+\ddot{U}_n\right)$$

The initial conditions are: $U_0 = 0$, $\dot{U}_0 = 0$, $\ddot{U}_0 = \left[M\right]^{-1}\left(R_0 - \left[K\right]U_0\right)$.

$$\ddot{U}_{n+1} = \frac{4}{\Delta t^2}\left[U_{n+1} - U_n\right] - \frac{4}{\Delta t}\dot{U}_n - \ddot{U}_n,$$

$$\dot{U}_{n+1} = \frac{\Delta t}{2}\left[\ddot{U}_{n+1} + \ddot{U}_n\right] + \dot{U}_n = \frac{2}{\Delta t}\left[U_{n+1} - U_n\right] - \dot{U}_n$$

% MATLAB Program:

```
% INITIAL VALUES
  M=[1 0;0 10];
  K=[21 -1;-1 1];
  C=[0.5 -0.1;-0.1 0.1];
  dt=0.05;
  T=2;dt=0.05;
  t=[0:dt:T];
  i=1;
  x(:,i)=[0;0];xd(:,i)=[0;0];
  f(:,i)=[0;10];
  xdd(:,i)=inv(M)*(f(:,i)-C*xd(:,i)-K*x(:,i));
  % FIRST time step
  dxd(:,2)=dt*xdd(:,1);
  for i=2:length(t)
  f(:,i)=[0;10];
  df(:,i)=f(:,i)-f(:,i-1);
  xd(:,i)=xd(:,i-1)+dxd(:,i);
  dx(:,i)=(dt/2)*(xd(:,i-1)+xd(:,i));
  dxdd(:,i)=inv(M)*(df(:,i)-K*dx(:,i)-C*dxd(:,i));
  xdd(:,i)=xdd(:,i-1)+dxdd(:,i);
  % UPDATING VALUES OF VELOCITY AND DISPLACMENT IN CURRENT CYCLE
    dxd(:,i)=(dt/2)*(xdd(:,i-1)+xdd(:,i));
    xd(:,i)=xd(:,i-1)+dxd(:,i);
    dx(:,i)=(1/2)*(xd(:,i-1)+xd(:,i));
    % REVISED DISPLACMENT IN CURRENT CYCLE
    x(:,i)=x(:,i-1)+dx(:,i);
    % DELTA x DOT FOR NEXT CYCLE
    dxd(:,i+1)=2*dt*xdd(:,i)-dxd(:,i);
  end
  fprintf('time\t\tX(1)\t\tX(2)\n');
  p=1
  for time=0:dt:T
    fprintf('%f\t%f\t%f\n',time,x(1,p),x(2,p));
    p=p+1;
  end
   plot(t,x(1,:),'-p',t,x(2,:),'-*');
      xlabel('time(s)');
   ylabel('displacement(m)');
   legend('DOF-1','DOF-2',2);
      grid on;
```

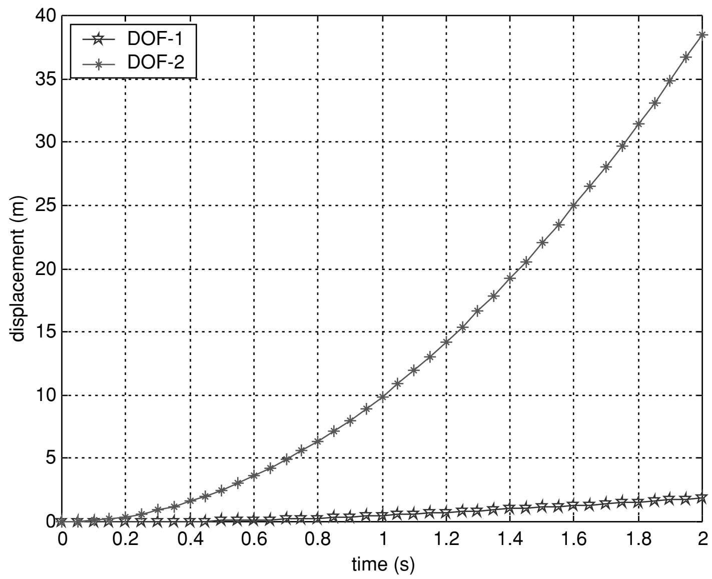


Figure AE.12 MATLAB output

Problem AE.13

Solve Problem AE.11 using the fourth-order Runge-Kutta method

Solution:

Here $\dot{Y} = f(x_1, x_2, x_3, x_4, t)$, where $Y = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$ and $f(x_1, x_2, x_3, x_4, t) = \begin{Bmatrix} x_3 \\ x_4 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix}$

Where $\begin{Bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = -[M]^{-1}[K]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} - [M]^{-1}[C]\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix} + [M]^{-1}F(t)$

In total it can be written as
$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} [0] \\ [M]^{-1}F(t) \end{bmatrix}$$

Or
$$\dot{Y} = [E]Y + F$$

% MATLAB Program:

```
dt=0.05;T=1;
h=dt;
x1=0; x2=0; % displacements
x3=0; x4=0; % velocity
M=[1 0;0 10];K=[21 -1;-1 1];C=[0.5 -0.1;-0.1 0.1];f=[0;10];
E=[zeros(size(M)) eye(size(M));-inv(M)*K -inv(M)*C];
F=[0;0;inv(M)*f];
i=1;
```

```
Y=[x1;x2;x3;x4];
fprintf('time\t\tX(1)\t\tX(2)\n');
for t=0:h:T
   X(:,i)=Y;
   fprintf('%f\t%f\t%f\n',t,Y(1),Y(2));
   K1=h*(E*Y+F);
   K2=h*(Y+ (0.5*K1))+F);
   K3=h*(Y+ (0.5*K2))+F)
   K4=h*(Y+K3+F)
   Y=Y+(K1+2*K2+2*K3+K4)/6;
   i=i+1;
end
time=[0:h:T];
plot(time,X(1,:),'-p',time,X(2,:),'-*');
grid on;
xlabel('time(s)');
ylabel('displacement(m)')
legend('DOF-1','DOF-2');
```

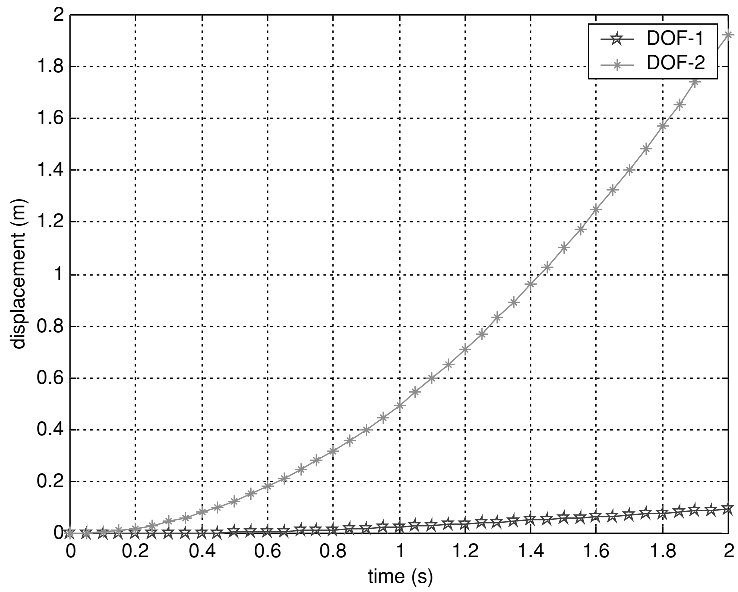


Figure AE.13 MATLAB output

PROBLEM AE.14

Solve Problem AE.11 using the Houbolt method.

Solution:

% MATLAB Program:

```
K=[21 -1;-1 1];
M=[1 0;0 10];
C=[0.5 -0.1;-0.1 0.1];
dt=0.05;T=2;
X0=[0;0];X0d=[0;0];F=[0;10];
t=[0:dt:T];
X(:,2)=X0;
X0dd=inv(M)*(F-C*X0d-K*X0);
% USING CENTRAL DIFFERENCE METHOD TO OBTAIN PREVIOUS 3 VALUES
```

```
Xprev=X0-(dt*X0d)+((dt^2)*(X0dd/2));
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=(a0*M)+(a1*C);
kbar=(K-a2*M);
cbar=(a0*M-a1*C);
X(:,1)=X0;
Fbar=F-kbar*X0-cbar*Xprev;
X(:,2)=inv(mbar)*Fbar;
Fbar=F-kbar*X(:,2)-cbar*X0;
X(:,3)=inv(mbar)*Fbar;

% HOUBOLT METHOD BEGINS
a0=2/(dt^2);a1=11/(6*dt);a2=5/(dt^2);a3=3/dt;a4=-2*a0;
a5=-a3/2;a6=a0/2;a7=a3/9;
Kb=K+a0*M+a1*C;
p=3;
for i=3:length(t)
   F=[0;10];% F(t+2dt)
   Fb=F+M*(a2*X(:,i)+a4*X(:,i-1)+a6*X(:,i-2))+ C*(a3*X(:,i)
      +a5*X(:,i-1)+a7*X(:,i-2));
   X(:,i+1)=inv(Kb)*Fb;
   Xdd(:,i+1)=a0*X(:,i+1)-a2*X(:,i)-a4*X(:,i-1)-a6*X(:,i-2);
   Xd(:,i+1)=a1*X(:,i+1)-a3*X(:,i)-a5*X(:,i-1)-a7*X(:,i-2);
   p=p+1;
end
fprintf('\ntime\t\tX1\t\tX2\n');
for i=1:p
   time(i)=(i-1)*dt;
   fprintf('%f\t%f\t%f\n',time(i),X(1,i),X(2,i))
end
plot(time,X(1,:),'-p',time,X(2,:),'-*');
grid on;
xlabel('Time(s)');
ylabel('displacement (m)');
legend('DOF-1','DOF-2');
```

Fig. AE.14 shows the plot of histories of two degrees of freedom.

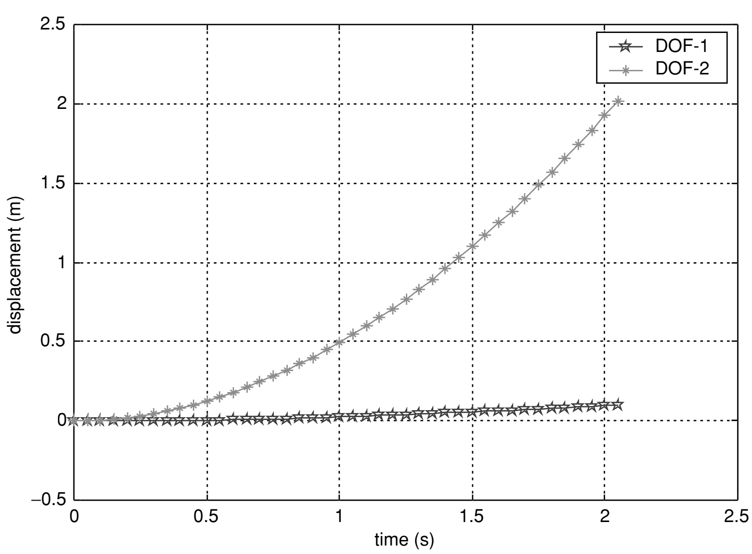


Figure AE.14 MATLAB output (Houbolt's method)

Problem AE.15

Solve Problem AE.11 using the Wilson Theta method.

Solution:

% MATLAB Program:

```
K=[21 -1;-1 1];
M=[1 0;0 10];
C=[0.5 -0.1;-0.1 0.1];
dt=0.05;T=2;
X0=[0;0];X0d=[0;0];F0=[0;10];
X0dd=inv(M)*(F0-C*X0d-K*X0);
theta=1.4;
a0=6/(theta*dt)^2;a1=3/(theta*dt);a2=2*a1;
a3=2;a4=(1/2*theta*dt);a5=-a2/theta;
a6=1-3/theta;
a7=dt/2;a8=dt^2/6;
Kb=K+a0*M+a1*C;
i=1;
X(:,1)=X0;Xd(:,1)=X0d;Xdd(:,1)=X0dd;t=0;
fprintf('time(s)\t\tX1\t\tX2\n');
fprintf('%f\t%f\t%f\n',t,X(1,1),X(2,1));
for t=dt:dt:T
   i=i+1;
   F=[0;10];
   Ftb=F0+M*(a0*X(:,i-1)+a2*Xd(:,i-1)+a3*Xdd(:,i-1))+C*
      (a1*X(:,i-1)+a3*Xd(:,i-1)+a4*Xdd(:,i-1))+theta*(F-F0);
   Xt(:,i)=inv(Kb)*Ftb;
   Xdd(:,i)=(a0/theta)*(Xt(:,i)-X(:,i-1))+a5*Xd(:,i-1)+a6*Xdd
      (:,i-1);
   Xd(:,i)=Xd(:,i-1)+a7*(Xdd(:,i)+Xdd(:,i-1));
   X(:,i)=X(:,i-1)+dt*Xd(:,i-1)+a8*(Xdd(:,i)+2*Xdd(:,i-1));
   F0=F;
   fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
end
t=[0:dt:T];
plot(t,X(1,:),'-p',t,X(2,:),'-*')
xlabel('Time(s)');
ylabel('displacement(m)');
legend('DOF-1','DOF-2');
grid on;
```

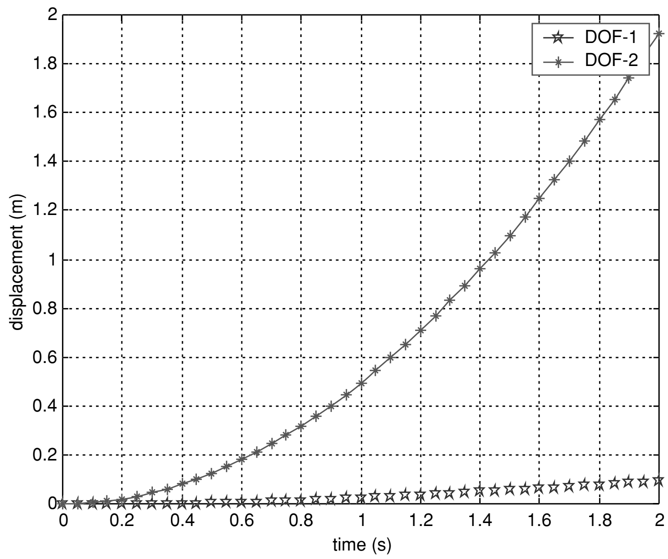


Figure AE.11 MATLAB response (Wilson-theta)

Problem AE.16

Solve Problem AE.11 using the Newmark Beta method.

Solution:

TABLE 1
THE NEWMARK SCHEME

- $u_{n+1} = u_n + hv_n + h^2(1/2 - \beta)a_n + h^2\beta a_{n+1}$
- $v_{n+1} = v_n + h\,(1 - \gamma)a_n + h\gamma a_{n+1}$
- $Ma_{n+1} + Cv_{n+1} + Ku_{n+1} = F_{n+1}$

Here, for $\beta = 0.5$ and $\gamma = 1/6$, the following values are obtained.

% MATLAB Program:

```
K= [21 -1;-1 1];
M= [1 0; 0 10];
C= [0.5 -0.1;-0.1 0.1];
dt=0.05;T=2;
X0=[0;0];X0d=[0;0];F=[0;10];
X0dd=inv(M)*(F-C*X0d-K*X0);
beta=0.5;gama=1/6;%0.25* (0.5+beta);
a0=1/(beta*dt^2);a1=gama/(beta*dt);a2=1/(beta*dt);
a3=(1/2*beta)-1;a4=(gama/beta-1);a5=0.5*(gama/beta-2)*dt;
a6=dt*(1-beta);a7=beta*dt;
Kb=K+a0*M+a1*C;
i=1;
X(:,1)=X0;Xd(:,1)=X0d;Xdd(:,1)=X0dd;t=0;
fprintf(‘time(s)\t\tX1\t\tX2\n’);
fprintf(‘%f\t%f\t%f\n’,t,X(1,1),X(2,1));
for t=dt:dt:T
   i=i+1;
   F=[0;10];
   Fb=F+M*(a0*X(:,i-1)+a2*Xd(:,i-1)+a3*Xdd(:,i-1))+C*(a1*X(:,
      i-1)+a4*Xd(:,i-1)+a5*Xdd(:,i-1));
   X(:,i)=inv(Kb)*Fb;
```

```
    Xdd(:,i)=a0*(X(:,i)-X(:,i-1))-a2*Xd(:,i-1)-a3*Xdd(:,i-1);
    Xd(:,i)=a1*(X(:,i)-X(:,i-1))-a4*Xd(:,i-1)-a5*Xdd(:,i-1);
    fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
end
t=[0:dt:T];
plot(t,X(1,:),'-p',t,X(2,:),'-*')
xlabel('Time(s)');
ylabel('displacement(m)');
legend('DOF-1','DOF-2');
grid on;
```

Fig. AE.16 shows the plot of histories of the two degrees of freedom.

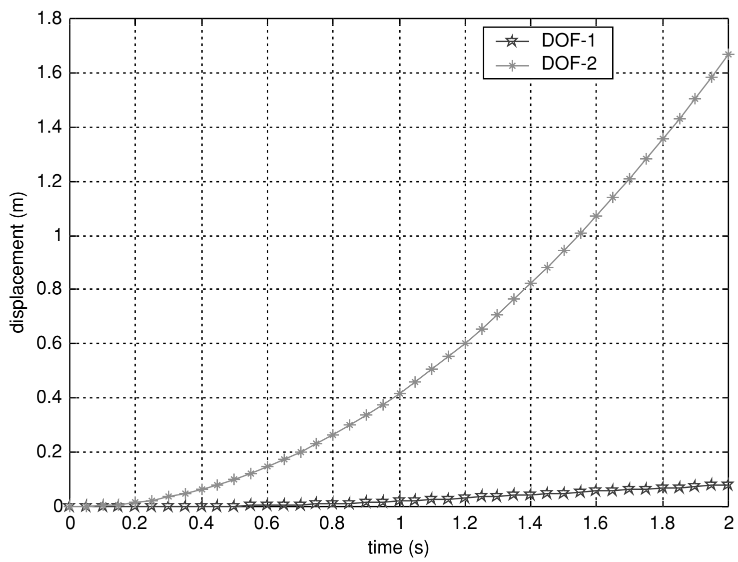


Figure AE.16 MATLAB output-Newmark Method

Problem AE.17

Solve Problem AE.11 using the Park Stiffly stable method.

Solution:

% MATLAB Program:

```
K=[21 -1;-1 1];
M=[1 0;0 10];
C=[0.5 -0.1;-0.1 0.1];
dt=0.05;T=2;
X0=[0;0];X0d=[0;0];F=[0;10];
t=[0:dt:T];
X(:,2)=X0;
X0dd=inv(M)*(F-C*X0d-K*X0);
% USING CENTRAL DIFFERENCE METHOD TO OBTAIN PREVIOUS 3 VALUES of
X AND Xd
Xprev=X0-(dt*X0d)+((dt^2)*(X0dd/2));
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=(a0*M)+(a1*C);
kbar=(K-a2*M);
cbar=(a0*M-a1*C);
X(:,1)=X0;Xd(:,1)=X0d;
```

```
Fbar=F-kbar*X0-cbar*Xprev;
X(:,2)=inv(mbar)*Fbar;
Fbar=F-kbar*X(:,2)-cbar*X0;
X(:,3)=inv(mbar)*Fbar;
Fbar=F-kbar*X(:,3)-cbar*X0;
X(:,4)=inv(mbar)*Fbar;
Xd(:,2)=a1*(X(:,3)-X(:,1));
Xd(:,3)=a1*(X(:,4)-X(:,2));

% PARK METHOD BEGINS
a0=10/(6*dt);a1=-15/(6*dt);a2=1/dt;a3=-1/(6*dt);
Kb=K+(a0^2)*M-a0*C;
p=3;
for i=3:length(t)
 F=[0;10];% F(t+3dt)
 mass=M*(-a1*Xd(:,i)-a2*Xd(:,i-1)-a3*Xd(:,i-2)-(a0*a1)*X(:,i)-
    (a0*a2)*X(:,i-1)+(a3^2)*X(:,i-2));
 damp=C*(a1*X(:,i)+a2*X(:,i-1)+a3*X(:,i-2));
 Fb=F+mass-damp;
 X(:,i+1)=inv(Kb)*Fb;
 Xd(:,i+1)=a0*X(:,i+1)+a1*X(:,i)+a2*X(:,i-1)+a3*X(:,i-2);
 Xdd(:,i+1)=a0*Xd(:,i+1)+a1*Xd(:,i)+a2*Xd(:,i-1)+a3*Xd(:,i-2);
 p=p+1;
end
fprintf('\ntime\t\tX1\t\tX2\n');
for i=1:p
  time(i)=(i-1)*dt;
    fprintf('%f\t%f\t%f\n',time(i),X(1,i),X(2,i))
end
plot(time,X(1,:),'-p',time,X(2,:),'-*');
grid on;
xlabel('Time(s)');
ylabel('displacement (m)');
legend('DOF-1','DOF-2');
```

Fig. AE.17 shows the MATLAB plot.

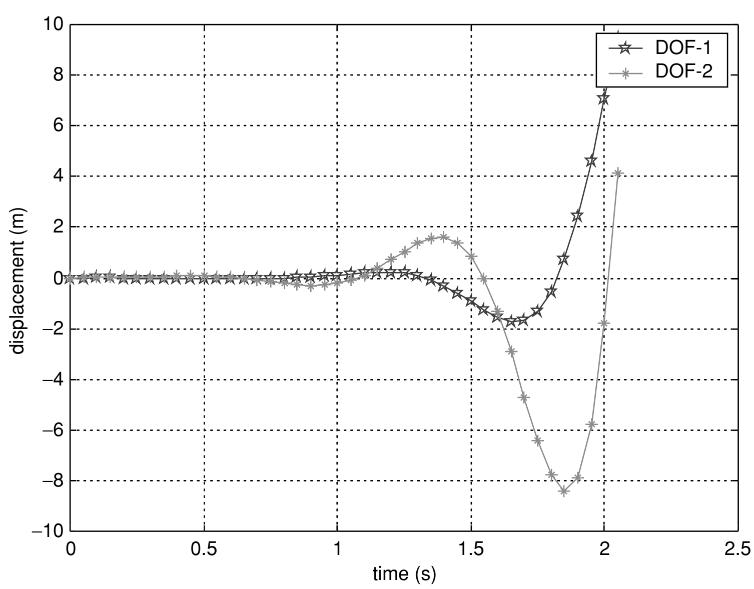


Figure AE.17 MATLAB output

Problem AE.18

The numerical values of the mass, damping, and stiffness are chosen as $M_1 = 1$, $M_2 = 10$, $C_1 = 0$, $C_2 = 0.15$, $K_1 = 19$ and $K_2 = 1$. The initial conditions are selected as zero and the forcing vector is

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix} \quad \text{for } t > 0$$

and

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{for } t < 0$$

In matrix motion, these equations may be written as

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} C_1 + C_2 & -C_2 \\ -C_2 & C_2 \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{Bmatrix} -0.5K_2(X_2 - X_1)^3 \\ 0.5K_2(X_2 - X_1)^3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Take $\Delta t = 0.05$ s. Use the central difference method and compute the response of the system.

Solution:

Substituting the given values, the following matrices is obtained:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 0.15 & -0.15 \\ -0.15 & 0.15 \end{bmatrix}, \quad K = \begin{bmatrix} 20 & -1 \\ -1 & 1 \end{bmatrix}, \quad F = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

% MATLAB Program:

% INITIAL VALUES

```
M=[1 0;0 10];
K=[20 -1;-1 1];
C=[0.15 -0.15;-0.15 0.15];
dt=0.05;
x0=[0;0];x0d=[0;0];
F0=[0;5];
T=2;
x0dd=inv(M)*(F0-C*x0d-K*x0);
xprev=x0-(dt*x0d)+((dt^2)*(x0dd/2));
a0=1/dt^2;a1=1/(2*dt);a2=2*a0;
mbar=(a0*M)+(a1*C);
 t=0;
 v(:,1)=x0d;a(:,1)=x0dd;
 i=1;
 fprintf('time\t\tX(1)\t\tX(2)\n');
 for t=0:dt:T+dt
 X(:,i)=x0;
```

```
% NONLINEAR SPRING
Fr=[-0.5*1*(X(2,i)-X(1,i))^3;0.5*1*(X(2,i)-X(1,i))^3];
F=F0+Fr;
Fbar=F+(a2*M-K)*x0+(a1*C-a0*M)*xprev;
x=inv(mbar)*Fbar;
xprev=x0;
x0=x;
fprintf('%f\t%f\t%f\n',t,X(1,i),X(2,i));
i=i+1;
p=i;
end
for i=2:p-1
if i<p-1
v(:,i)=(X(:,i+1)-X(:,i-1))*(1/(2*dt));
a(:,i)=(X(:,i+1)-2*X(:,i)+X(:,i-1))*(1/dt^2);
end
end
t=[0:dt:T+dt];
plot(t,X(1,:),'-p',t,X(2,:),'-*');
xlabel('time(s)');
ylabel('displacement(m)');
legend('DOF-1','DOF-2',2);
grid on;
```

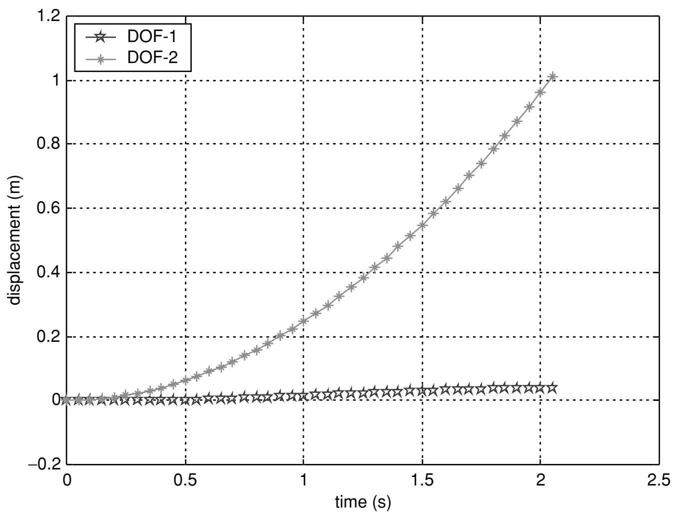


Figure AE.18 MATLAB output

PROBLEM AE.19

Consider a simple system for which the governing equilibrium equations are

$$[M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$F_1(t) = 0, \quad F_2(t) = 10$$

(a) calculate the transformation matrix for this system and thus establish the decoupled equations of equilibrium on the basis of mode shape vectors.

(b) compute the exact response by integrating each of the two decoupled equilibrium equations.

Solution:

(a) Transformation matrix of the system is obtained from eigenvalue problem.

Solving the undamped eigenvalue equation, we obtain eigenvector as

$$X = \begin{bmatrix} -0.8069 & 0.3437 \\ -0.5907 & -0.9391 \end{bmatrix} \text{corresponding to eigenvalues } \lambda = \begin{bmatrix} 1.634 \\ 3.366 \end{bmatrix}$$

Find modal masses $m_1 = \text{sqrt}(X_1'MX_1) = 1.2850$

$$m_2 = \text{sqrt}(X_2'MX_2) = 1.0574$$

Now the transformation matrix is

$$\phi = \begin{bmatrix} -0.8069/1.2850 & 0.3437/1.0574 \\ -0.5907/1.2850 & -0.9391/1.0574 \end{bmatrix} = \begin{bmatrix} -0.628 & 0.325 \\ -0.4597 & -0.8881 \end{bmatrix}$$

$$\phi'F = \begin{bmatrix} -0.628 & 0.325 \\ -0.4597 & -0.8881 \end{bmatrix}' \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} -4.597 \\ -8.8808 \end{bmatrix}$$

Thus, the decoupled equations are

$$[\phi'M\phi]\ddot{U} + [\phi'K\,\phi]\,U = \phi'F$$

That is modal equations are:

$$\ddot{u}_1 + 1.634\,u_1 = -4.597$$

$$\ddot{u}_2 + 3.366\,u_2 = -8.8808$$

(b) Each of these equations is solved independently by integration as follows:

Solution is u = C.F + P.I.,

where C.F. = complementary function = $e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$ and

$$\text{P.I.} = \text{Particular integral} = f/\omega_n^2$$

The constants A and B are obtained with initial condition $u(0)$ and $\dot{u}(0)$

$$u(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + f/\omega_n^2$$

$$\dot{u} = e^{-\xi\omega_n t}((B\omega_d - \xi\omega_n A)\cos\omega_d t - (A\omega_d + \xi\omega_n B)\sin\omega_d t)$$

Hence, $\quad u(0) = A + f/\omega_n^2 \quad$ or $\quad A = u(0) - f/\omega_n^2$

and $\quad \dot{u}(0) = (B\omega_d - \xi\omega_n A) \quad$ or $\quad B = [\dot{u}(0) + \xi\omega_n A]/\omega_d$

If there is no damping $\xi = 0$, thus $B = \dot{u}(0)/\omega_n$

Also with zero initial conditions, we can write the solutions as follows:

$$u_1 = f_1/\omega_{n1}^2(1 - \cos\omega_{n1}t) \quad \text{and} \quad u_2 = f_2/\omega_{n2}^2(1 - \cos\omega_{n2}t)$$

or $u_1 = [-4.597/1.634]\,(1 - \cos 1.2783t)$ and $u_2 = [-8.8808/3.366]\,(1 - \cos 1.8347t)$

$= -2.8133\,(1 - \cos 1.2783t)$ $\qquad = -2.6384\,(1 - \cos 1.8347t)$

Finally, the response in original coordinates is obtained as follows:

$$X = \phi U = \begin{bmatrix} -0.628 & 0.325 \\ -0.4597 & -0.8881 \end{bmatrix} \begin{bmatrix} -2.8133(1 - \cos 1.2783t) \\ -2.6384(1 - \cos 1.8347t) \end{bmatrix}$$

$$= \begin{bmatrix} 0.9091(1 - \cos 1.2783t) \\ 3.6364(1 - \cos 1.8347t) \end{bmatrix}$$

% MATLAB Program:

```
M=[2 0; 0 1];
K=[4 -1;-1 3];
F=[0;10];
[u,W]=eig(K,M);
for i=1:2
  wn(i)=sqrt(W(i,i));
end
[w,I]=sort(wn);
for j=1:2
U(:,j)=u(:,I(j)); % ARRANGE MODAL VECTORS IN ASCENDING ORDER
end
% MODAL MASSES
m1=sqrt(U(:,1)'*M*U(:,1));
m2=sqrt(U(:,2)'*M*U(:,2));
phi=[U(:,1)/m1;U(:,2)/m2]; % transformation matrix

% PLOT RESPONSES
t=[0:0.1:5];
u1=-2.8133*(1-cos(1.2783*t)); u2=-2.6384* (1-cos (1.8347*t));
x1=0.9091*(1- cos(1.2783*t)); x2=3.6364* (1-cos (1.8347*t));
subplot(2,1,1);                    %SUB-PLOT-1
plot(t,u1,'-p',t,u2,'-*');
title('\bfPlot of Modal amplitudes');
xlabel('\bfTime(s)');
legend('At frequency-1=1.2783 rad/s','At frequency-2=1.8347 rad/s');
grid on;
subplot(2,1,2);                    % SUB-PLOT-2
plot(t,x1,'-p',t,x2,'-*');
title('\bfPlot of Original amplitudes');
xlabel('\bfTime(s)');
legend('At DOF-1', 'At DOF-2');
grid on;
```

The output of the program is shown in Fig. AE.19.

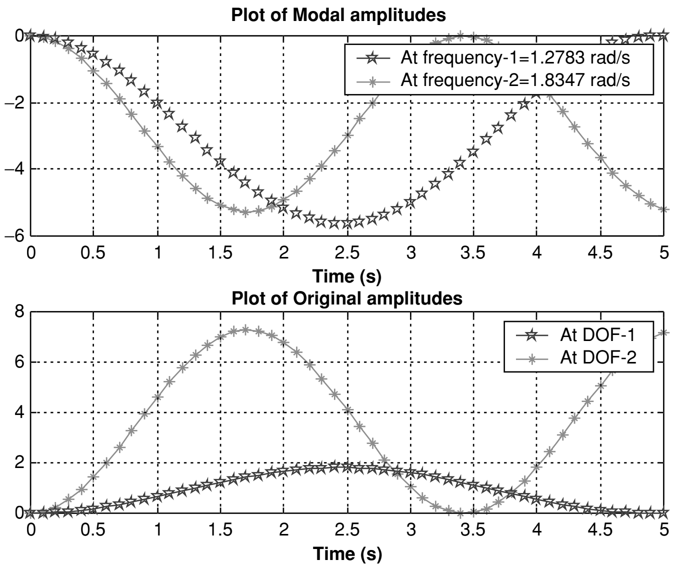


Figure AE.19 Output of the MATLAB program

Summary

We have briefly reviewed the direct numerical integration methods for the solution of a single or system of differential equations. Many numerical methods are available for the solutions of the response of dynamic systems. We have discussed several widely used step-by-step numerical integration methods for linear dynamic response analysis. A brief description of these integration methods is presented and their application is illustrated. The integration schemes considered are three explicit and four implicit methods. They are the explicit schemes (the central difference method, two-cycle interaction with trapezoidal rule, and fourth-order Runge-Kutta method) and the implicit schemes (Houbolt method, Wilson Theta method, Newmark Beta method, and the Park Stiffly stable method). Application of these direct numerical integration methods is illustrated with numerical examples for a linear dynamic system. The use of a particular integration method is mainly dependent on the nature of the problem and is often dictated by the desired solution accuracy.

AutoLibrary

Appendix F

Units and Conversions

Problem F.1

Express torque, T, (SI) in US/English units.

Express spring constant, k, (SI) and mass (M) in US/English units

$$\begin{aligned}
[\text{Torque in SI units}] &= [\text{Torque in English units}] \times [\text{multiplying factors}] \\
[\text{N. m}] &= [\text{lb. in.}]\,[\text{N/lb}]\,[\text{m/in.}] \\
&= [\text{lb. in.}]\,[4.448]\,[0.0254] \\
&= [\text{lb. in.}]\,[0.1129]
\end{aligned}$$

Problem F.2

Express mass moment of inertia, I, (SI) in US/English units.

Mass moment of inertia = Mass moment of inertia × multiplication factor

$$\begin{aligned}
(\text{kg. m}^2) = (\text{N. m. s}^2) &= [\text{N per 1 lb}_f]\,[\text{m per 1 in.}]\,[\text{lb}_f - \text{in.} - \text{s.}^2] \\
&= [4.448222]\,[0.0254]\,[\text{lbf} - \text{in.} - \text{s.}^2] \\
&= [0.1129848]\,[\text{lb}_f - \text{in.} - \text{s.}^2]
\end{aligned}$$

Problem F.3

Express modulus of elasticity, E, (SI) in US/English units.

Modulus of elasticity, E:

$$\left[\frac{\text{N}}{\text{m}^2}\right] = \left[\frac{\text{lb}}{\text{in.}^2}\right]\left[\frac{\text{N}}{\text{lb}}\right]\left[\frac{\text{in.}^2}{\text{m}}\right]$$

E of steel $\text{N/m}^2 = (29 \times 10^6 \text{ lb/in.}^2)\,(6894.7) = 200 \times 10^9 \text{ N/m}^2$

Problem F.4

Express mass moment of inertia, I, (SI) in US/English units.

Spring stiffness, k:

$$[\mathrm{N/m^2}] = [\mathrm{lb/in.}] \times (175.13)$$

Mass, m:

$$[\mathrm{kg}] = [\mathrm{lb.\ s^2/in.}] \times (175.13)$$

Appendix G

Accident Reconstruction Formulae

Problem G.1

a) A dump truck backs over and kills a laborer. The truck moved 87 ft [26.5 m] at an average speed of 5 mph [8 kph] while it was backing. How much time elapsed between the beginning of the backup and the accident?

b) A vehicle is traveling at a constant speed of 82 mph [132 km/h]. How far will it travel in 2 s?

Solution:

a) To calculate the distance covered, we must first convert speed in miles per hour to velocity in feet per second. The variables:

S = Speed = 5 mph

V = Velocity in feet/second

d = distance = 87 ft

t = time

The conversion factor is 1.467 ft/s = 1 mph. Multiplying that into 5 mph yields velocity:

$$V = S*1.467 = 5*1.467 = 7.33 \text{ ft/s}.$$

Total distance

$$t = d/V = 87*7.33 = 11.9 \text{ s}.$$

b) To calculate the distance covered, we must first convert speed in miles per hour to velocity in feet per second. The variables:

S = Speed = 82 mph

V = Velocity in feet/second

t = time = 2 s

d = distance

The conversion factor is 1.467 ft/s = 1 mph. Multiplying that into 40 mph yields velocity:

$$V = S*1.467 = 82*1.467 = 120.3 \text{ ft/s}$$

Total distance

$$d = V*t = 120.3*2 = 241 \text{ ft } [73.3 \text{ m}]$$

Problem G.2

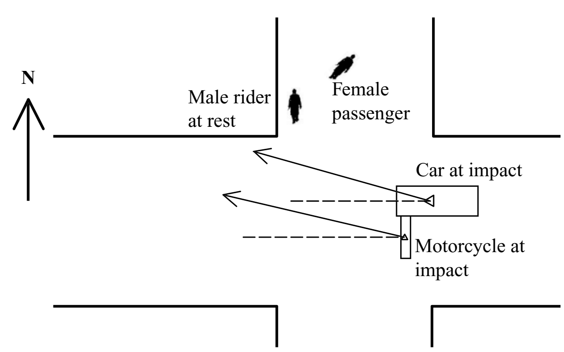
a) A car moves 15 ft [4.6 m] after colliding with another vehicle. The average stopping drag factor is 0.50. What is the post crash speed of the car?

b) A vehicle is traveling at a constant speed of 40 mph [64 km/h]. How far will it travel in 5 s?

c) A car is traveling 75 mph [121 km/h] in a work zone on an interstate highway. A dump truck starts to pull into the car's lane. How much distance will be required for the driver of the car to react and bring his vehicle to a stop? Use 0.75 for the stopping drag factor and 1.1 s for reaction time.

d) Southbound 3600-lb [1633 kg] V-1 collides head-on with northbound 4350-lb [1973 kg] V-2. Both vehicles move north after impact. The delta V of V-2 was 23 mph [37 km/h]. Assuming no restitution, calculate the delta V experienced by V-1.

e) A motorcycle (V-1) collides at an intersection with an automobile (V-2). The motorcycle riders are ejected onto/over the hood/windshield of the auto and both are killed. The motorcycle slides on its side to a stop after impact. The auto spins out 55 ft [16.8 m], strikes a concrete curb and then travels another 10 ft [3.1 m] before coming to rest. Given the following data, calculate the impact speed of both vehicles. To simplify calculations, assume both riders have the same speed/direction as the motorcycle, post crash. You may also ignore the car's speed loss from the curb impact.



	V-1	V-2
Curb weight:	750 lb [340 kg]	3418 lb [1550 kg]
Driver weight:	205 lb [93 kg]	165 lb [75 kg]
Passenger weight:	140 lb [29 kg]	--------
Cargo weight:	-------	50 lb [23 kg]
Direction at impact:	North	West
Direction after impact:	W 20° N	W 25° N
Distance after impact:	80 ft [24 m]	65 ft [20 m]
Drag factor after impact:	0.50	0.58

f) Recalculate Problem (e) using the following data:

Male Motorcycle Rider:

Direction after impact	W 50° N
Post crash slide/tumble dist.:	36 ft [11 m]
Post crash drag factor:	1.00

Female Motorcycle Rider:

Direction after impact	W 60° N
Post crash slide/tumble dist.:	38 ft [11.5 m]
Post crash drag factor:	1.00
Equiv. Barrier speed loss of V-2's curb strike:	10 mph [16 km/h]

g) What if the motorcycle had a 5 degree evasive steer to the left at impact? How would that affect your estimate of the speed of the car?

Solution:

a) We will use the basic skid formula to solve this problem. The variables:

s = Speed of vehicle after impact

d = Post impact travel distance = 15 ft

f = Post impact drag factor = 0.50

Substituting and solving:

$$s = \sqrt{30 * d * t} = \sqrt{30 * 15 * 0.50}$$

$$s = \sqrt{225} = 15 \text{ mph [24 km/h]}$$

b) To calculate the distance covered, we must first convert speed in miles per hour to velocity in feet per second The variable:

S = Speed = 40 mph

V = Velocity in feet/second

t = time = 5 s

d = distance

The conversion factor is 1.467 ft/s = 1 mph. Multiplying that into 40 mph yields velocity:

$$V = S*1.467 = 40*1.467 = 58.7 \text{ ft/s}$$

Total distance:

$$d = V*t = 58.7*5 = 293 \text{ ft [89 m]}$$

c) To calculate the distance covered during the reaction phase, we must first convert speed in miles per hour to velocity in feet per second, then multiply velocity by reaction time. We then add that distance to required brake-to-stop distance. The variables:

S = Speed = 75 mph

V = Velocity in feet/sec

f = braking drag factor = 0.75

t = reaction time = 1.1 s

d = total distance

d_R = distance covered during reaction phase

d_B = distance required to brake to stop

The conversion factor is 1.467 ft/s = 1 mph. Multiplying that into 75 mph yields velocity:

$$V = S*1.467 = 75*1.467 = 110 \text{ ft/s}$$

Reaction distance:

$$d_R = V*t = 110*1.1 = 121 \text{ ft}$$

We use a form of the basic skid to determine the brake-to-stop distance:

$$d_B = S^2/(30*f) = 75^2/(30*0.75) = 250 \text{ ft}$$

The total is the sum of the reaction distance and the brake-to-stop distance:

$$d = d_R + d_B = 121 + 250 = 371 \text{ ft [113 m]}$$

d) This problem is simpler than it first appears. Delta V's are in inverse proportion to the vehicle weights. The variables:

ΔV_1 = Speed change of V-1 from impact

ΔV_2 = Speed change of V-1 from impact = 23 mph

W_1 = Weight of V-1 = 4350 lb

W_2 = Weight of V-2 = 3600 lb

The equation:

$\Delta V_1/\Delta V_2 = W_2/W_1$

$\Delta V_1/23 = 4350/3600 = 1.208$

$\Delta V_1 = 1.208*23 = 28.8$ mph [46 km/h]

e) We have a conservation of momentum problem on our hands. The variables:

S = Speed of V-2 at start of precrash braking

S_1 = Speed of V-1 at impact

s_1 = Speed of V-1 after impact

S_2 = Speed of V-2 at impact

s_2 = Speed of V-2 after impact

W_1 = Weight of V-1 = 750 + 205 + 140 = 1095 lb

W_2 = Weight of V-2 = 3418 + 165 + 50 = 3633 lb

d_1 = Post impact travel distance of V-1 = 80 ft

d_2 = Post impact travel distance of V-2 = 65 ft

f_1 = Post impact drag factor of V-1 = 0.50

f_2 = Post impact drag factor of V-2 = 0.58

α_1 = Travel angle of vehicle one at impact

α_2 = Travel angle of vehicle two at impact

β_1 = Travel angle of vehicle one after impact

β_2 = Travel angle of vehicle two after impact

We will start setting by assuming that West is our 0 degree line. All angles measured clockwise will be considered positive. We will tabulate our angles and the determine the trigonometric values:

	Direction	θ	**sin** θ	**cos** θ
α_1:	North	90°	1	0
α_2:	West	0°	0	1
β_1:	W 20° N	20°	0.342	0.940
β_2:	W 25° N	25°	0.423	0.906

The weight ratios: If $W_1 = 1$, $W_2 = 3633/1095 = 3.32$

Post impact speed of V-1:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 80 * 0.50} = 34.6 \text{ mph}$$

Post impact speed of V-2:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 65 * 0.58} = 33.6 \text{ mph}$$

The conservation of momentum formula along north/south axis:

$$S_1 * W_1 * \sin \alpha_1 + S_2 * W_2 * \sin \alpha_2 = s_1 * W_1 * \sin \beta_1 + s_2 * W_2 * \sin \beta_2$$

$$S_1 * 1 * 1 + S_2 * 3.32 * 0 = 34.61 * 1 * 0.342 + 33.6 * 3.32 * 0.423$$

$$S_1 = 11.8 + 47.2$$

$$S1 = 59.0 \text{ or } 59 \text{ mph } [95 \text{ km/h}]$$

The conservation of momentum along the east/west axis:

$$S_1 * W_1 * \cos \alpha_1 + S_2 * W_2 * \cos \alpha_2 = s_1 * W_1 * \cos \beta_1 + s_2 * W_2 * \cos \beta_2$$

$$59.0 * 1 * 0 + S_2 * 3.32 * 1 = 34.6 * 1 * 0.940 + 33.6 * 3.32 * 0.906$$

$$3.32 * S_2 = 32.5 + 101.1 = 133.6$$

$$S_2 = 40.2 \text{ or } 40 \text{ mph } [65 \text{ km/h}]$$

f) We have a more complex conservation of momentum problem on our hands. V-1 essentially breaks into three pieces after impact. Additional variables:

s_1 = Speed of V-1 (bike by itself) after impact

s_{IM} = Speed of V-1 male rider after impact

s_{IF} = Speed of V-1 female rider after impact

W_1 = Weight of V-1 (bike by itself) = 750 lb

W_{IM} = Weight of V-1 male rider = 205 lb

W_{IF} = Weight of V-1 female rider = 140 lb

d_{IM} = Post impact travel distance of V-1 male rider = 36 ft

d_{IF} = Post impact travel distance of V-1 female rider = 38 ft

f_{IM} = Post impact drag factor of V-1 male rider = 1.00

f_{IF} = Post impact drag factor of V-1 female rider = 1.00

β_{IM} = Travel angle of vehicle one male rider after impact

β_{IF} = Travel angle of vehicle one female rider after impact

Additional angles and trigonometric values:

	Direction	**θ**	**sin θ**	**cos θ**
α_{IM}:	W 50° N	50°	0.766	0.643
α_{IF}:	W 60° N	60°	0.866	0.500

The weight ratios:

If weight of V-1 (with riders) = 1, $W_1 = 750/1095 = 0.685$

If weight of V-1 (with riders) = 1, $W_{IM} = 205/1095 = 0.187$

If weight of V-1 (with riders) = 1, $W_{IF} = 140/1095 = 0.128$

Post impact speed of V-2 (accounting for 10 mph [ebs] curb strike):

$$s_2 = \sqrt{30 * 65 * 0.58 + 10^2} = 35.1\,\text{mph}$$

Post impact speed of V-1 male rider:

$$s_1 = \sqrt{30 * d_{IM} * f_{IM}} = \sqrt{30 * 36 * 1.0} = 35.9\,\text{mph}$$

Post impact speed of V-1 female rider:

$$s_1 = \sqrt{30 * d_{IF} * f_{IF}} = \sqrt{30 * 38 * 1.0} = 33.8\,\text{mph}$$

The conservation of momentum formula along north/south axis:

$$S_1*(W_1 + W_{IM} + W_{IF})*\sin\alpha_1 + S_2*W_2*\sin\alpha_2 = s_1*W_1*\sin\beta_1 + s_{IM}*W_{IM}*\sin\beta_{IM} + s_{IF}*W_{IF}*\sin\beta_{IF} + s_2*W_2*\sin\beta_2$$

$$S_1*1*1 + S_2*3.32*0 = 34.6*0.342 + 32.9*0.187*0.766 + 33.8*0.128*0.866 + 35.1*3.32*0.423$$

$$S_1 = 8.1 + 4.7 + 3.7 + 49.3$$

S_1 = 65.8 or 66 mph [106 km/h]

The conservation of momentum along the east/west axis:

$$S_1*(W_1 + W_{IM} + W_{IF})*\cos\alpha_1 + S_2*W_2*\cos\alpha_2 = s_1*W_1*\cos\beta_1 + s_{IM}*W_{IM}*\cos\beta_{IM} + s_{IF}*W_{IF}*\cos\beta_{IF} + s_2*W_2*\cos\beta_2$$

$$63.7*1*0 + S_2*3.32*1 = 34.6*0.685*0.940 + 32.9*0.187*0.643 + 33.8*0.128*0.500 + 35.1*3.32*0.906$$

$$3.32*S_2 = 22.4 + 4.0 + 2.2 + 105.6 = 134.2$$

S_2 = 40.4 or 40 mph [66 km/h]

The results of these problems show again that in cases in which a light vehicle collides with a much heavier vehicle, the light vehicle's calculated impact speed will be highly sensitive to changes in the variable. Conversely, the heavy vehicle's calculated impact speed will be relatively insensitive to changes in the variables.

g) This will take away from the speed of the car because some of the westerly post crash momentum, now attributed totally to the car precrash, must now be attributed to the bike precrash. However, because the bike is so much lighter than the car, this will reduce the car's precrash speed by only about 1 mph.

Problem G.3

a) A truck is traveling at 57 mph [92 kph]. A car is traveling in the opposite direction at 72 mph [116 kph]. The distance between them is 467 ft [142 m]. If they both maintain constant speed, how much time will elapse before they collide?

b) V-1 stops at an intersection. It then accelerates at a moderate rate of 0.25 g's for 35 ft [10.7 m], and is then struck broadside by another vehicle. How fast was V-1 going at impact?

Solution:

a) First, we must convert the speeds of the vehicles from miles per hour to velocities feet per second. The truck was travelling 57 mph and the car was traveling 72 mph.

Velocity of the truck: $V_T = 57*1.467 = 83.6$ ft/s

Velocity of the car: $V_C = 72*1.467 = 105.6$ ft/s

The closing velocity is the sum of the two vehicle velocities:

$$V = V_T + V_C = 83.6 + 105.6 = 189.2 \text{ ft/s}$$

The time required to cover the 467-ft distance is the distance divided by the closing velocity:

$$t = d/V = 467/189.2 = 2.5 \text{ s}$$

b) To calculate the average, we ultimately need to utilize the equation that gives the relation between acceleration, initial velocity, and distance.

$$d = d_O + V_O*t + 0.5*a*t^2$$

where: d = Total distance = 35 ft

d_O = Starting point = 0 ft

V_O = Initial velocity = 0 ft/s

a = acceleration rate, ft/s/s

t = time

First we convert acceleration from g's to ft/s/s:

$$a = 0.25*g = 0.25*32.2 = 8.05 \text{ ft/s/s}$$

Substituting and solving to find elapsed time:

$$35 = 0 + 0*t + 0.5*8.05*t^2$$

$$35 = 4.025*t^2$$

$$8.64 = t^2$$

$$2.95 \text{ s} = t$$

Velocity at the end of the test can be estimated by multiplying the acceleration by the time elapsed:

$$V = a*t = 8.05*2.95 = 23.7 \text{ ft/s}$$

Converting to miles per hour:

$$S = V/1.467 = 23.7/1.467 = 16.2 \text{ or } 16 \text{ mph [26 kph]}$$

Problem G.4

a) A truck is traveling at 47 mph [75.6 kph]. A car is traveling in the opposite direction at 60 mph [96.6 kph]. The distance between them is 367 ft [112 m]. If they both maintain constant speed, how much time will elapse before they collide?

b) V-l stops at an intersection. It then accelerates at a moderate rate of 0.25 g's for 42 ft [12.8 m], and is then struck broadside by another vehicle. How fast was V-l going at impact?

Solution:

a) First, we must convert the speeds of the vehicles from miles per hour to velocities feet per second. The truck was traveling 47 mph and the car was travelling 60 mph.

Velocity of the truck: $V_T = 47*1.467 = 68.9$ ft/s

Velocity of the car: $V_C = 60*1.467 = 88.0$ ft/s

The closing velocity is the sum of the two vehicle velocities:

$$V = V_T + V_C = 68.9 + 88.0 = 156.9 \text{ fl/s}$$

The time required to cover the 367-ft distance is the distance divided by the closing velocity:

$$t = d/V = 367/156.9 = 2.3 \text{ s}$$

b) To calculate the average, we ultimately need to utilize the equation that gives the relation between acceleration, initial velocity and distance.

$$d = d_O + V_O*t + 0.5*a*t^2$$

where d = Total distance = 42 ft

d_O = Starting point = 0 ft

V_O = Initial velocity = 0 ft/s

a = acceleration rate, ft/s/s

t = time

First we convert acceleration from g's to ft/s/s:

$$a = 0.25*g = 0.25*32.2 = 8.05 \text{ ft/s/s}$$

Substituting and solving to find elapsed time:

$$42 = 0 + 0*t + 0.5*8.05*t^2$$

$$42 = 4.025*t^2$$

$$10.43 = t^2$$

$$3.23 \text{ s} = t$$

Velocity at the end of the test can be estimated by multiplying the acceleration by the time elapsed:

$$V = a * t = 8.05 * 3.23 = 26.0 \text{ ft/s}$$

Converting to miles per hour:

$$S = V/1.467 = 26.0/1.467 = 17.7 \text{ or } 18 \text{ mph [28 kph]}$$

Problem G.5

a) A motorcycle skids 55 ft [16.8m] with only the rear wheel skidding (drag factor = 0.35). It then skids 34 ft [10.4 m] with both wheels sliding (drag factor = 0.88). It then slides 83 ft [25.3 m] on its side (drag factor = 0.52). Determine the speed of the bike at the start of the one-wheel skid.

b) V-l is travelling 32 mph [51.5 kph] when it hits another car. V-l Skidded 41 ft [12.5 m] at a drag factor of 0.77 just prior to impact. What was the speed of V-l at the start of skid?

Solution:

We have a combined speed problem on our hands. The variables:

S = Speed of motorcycle at start of precrash skid

d_1 = One-wheel skid distance = 55 ft

f_1 = One-wheel skid drag factor = 0.35

d_2 = Two-wheel skid distance = 34 ft

f_2 = Two-wheel skid drag factor = 0.88

d_s = Side slide distance = 83 ft

f_s = Side slide drag factor = 0.52

The speed of V-l at the start of precrash braking:

$$S = \sqrt{30 * d_1 * f_1 + 30 * d_2 * f_2 + 30 * d_s * f_s}$$

$$S = \sqrt{30 * 55 * 0.35 + 30 * 34 * 0.88 + 30 * 83 * 0.52}$$

$$S = \sqrt{57735 + 879.6 + 1294.8} = \sqrt{2769.9}$$

$$S = 52.63 \text{ or } 53 \text{ mph [85 km/h]}$$

This problem is solved by using the combined speed formula. The variables:

S_1 = Speed of V-1 at impact = 32 mph

d = Pre-impact skid distance = 41 ft

f = Drag factor of truck = 0.79

$$S = \sqrt{S_1^2 + 30 * d * f}$$

$$S = \sqrt{32^2 + 30 * 41 * 0.79}$$

$$S = \sqrt{1024 + 972} = \sqrt{1996}$$

$$S = 44.67 \text{ or } 45 \text{ mph [72 kph]}$$

Problem G.6

A northbound train is stopped on a siding with its last car a short distance from a highway grade crossing, as shown in Fig. G.6. A southbound train is approaching the crossing but is screened by the other train. Meanwhile, two eastbound autos are stopped for the flashing crossing light and gate. A third auto stops behind the other two. The third car accelerates around the other two cars, attempts to maneuver around the crossing the crossing gates, and is struck by the southbound train.

a) The car accelerated from a stop to 25 mph [40.2 km/h], and maintained constant speed until struck by the train. If its acceleration rate was 0.30g, how much time and how much distance was required for it to reach 25 mph.

b) The car traveled 103 ft [31.4 m] from where it was stopped to where it was struck. How much time did it require to travel this distance.

c) The southbound train was traveling at a constant speed of 62 mph [100 km/h]. How far from the point of impact was it when the car started accelerating?

d) The car needed to travel an additional 12 ft [3.7 m] to avoid the train. At 6.5 mph, how many more seconds would this require?

e) After hitting the car, the train takes 5075 ft [1547 m] to stop. What was me average post impact drag factor of the train.

f) What was the kinetic energy of the 3280-lb [1488 kg] car at 25 mph? What was the kinetic energy of the entire train at 62 mph if the train has a total weight of 5000 tons?

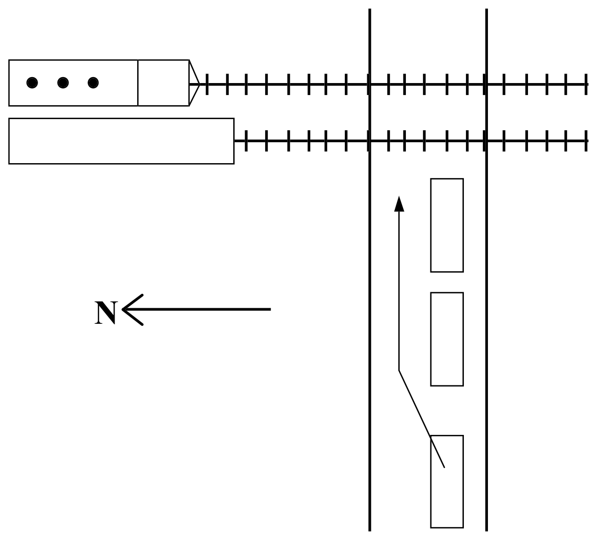


Figure G.6

Solution:

a) First, we must find the velocity of the car (after acceleration) in feet per sec:

$$V = 25*1.467 = 36.7 \text{ ft/s}$$

We know that one 'g' equals 32.2 ft/s/s, an acceleration of 0.3 g equals:

$$0.3*g = 0.3*32.2 = 9.66 \text{ ft/s/s}$$

The time required to accelerate to 25 mph:

$$t = V/a = 36.7/9.66 = 3.8 \text{ s}$$

The distance required to accelerate to 25 mph:

$$d = V_O{*}t + 0.5{*}a{*}t^2 = 0{*}3.8 + 0.5{*}9.66{*}3.8^2$$

$$d = 0 + 0.5{*}9.66{*}14.4 = 69.7 \text{ feet [21.3 m]}$$

b) We know from Problem 1 how much time/distance it took for the car to hit 25 mph. Now we must determine how much additional time/distance it took to reach point of impact. Total distance from start to point of impact (d_T) was 103 ft.

The distance traveled at a constant 25 mph (d_C) was:

$$d_C = d_T - d = 103 - 69.7 = 33.3$$

Time required covering the final 33.3 ft:

$$t = d_C/V = 33.3/36.7 = 0.9 \text{ s}$$

Total elapsed Time (t_T):

$$t_T = 3.8 + 0.9 = 4.7 \text{ s}$$

c) We have another time/distance problem on our hands. We know from Problem 1 that the time from when the car starts accelerating until it reaches the point of impact is 4.7 s. We also know the train's speed (S_T) equals 62 mph. If we next determine the train's velocity (V_T) in feet per second we can the distance cover in that 4.7 s. timeframe:

$$V_T = S_T{*}1.467 = 62{*}1.467 = 91.0 \text{ ft/s}$$

$$d_T = V{*}t = 91.0{*}4.7 = 428 \text{ feet}$$

d) From Problem 1, we learned a constant 25 mph converts to 36.7 ft/s. The time required to go 12 ft:

$$t = d/V = 12/36.7 = 0.33 \text{ s}$$

e) We can find the train's average drag factor by simply using a variant of the basic skid formula:

$$f = S^2/(30{*}d)$$

We can find the train's average drag factor by simply using a variant of the basic skid formula:

$$f = 62^2/(30{*}5075) = 3844/152250 = 0.025$$

f) The same method is used for both the train and the car. To determine a moving object's kinetic energy, we use the classic physics formula:

$$KE = 0.5{*}M{*}V^2 = 0.5{*}(W/g){*}V^2$$

where KE = Kinetic Energy (ft*lb)

M = Mass of the object = (lb*s^2/ft)

V = Velocity of the object (ft/sec) (36.7 ft/s for car; 91.0 ft/s for train)

W = Weight of the object (5000 tons*2000 lb/ton = 10,000,000 lb for train)

g = Acceleration of gravity = 32.2 ft/s^2

Calculating for the car (weight = 3280 lb):

$$KE = 0.5*(W/g)*V^2 = 0.5*3280*36.7^2 = 2{,}210{,}000 \text{ ft*lb}$$

Calculating for the train:

$$KE = 0.5*(W/g)*V^2 = 0.5*10{,}000{,}000*98.3^2 = 48{,}300{,}000{,}000 \text{ ft*lb}$$

Problem G.7

V-l is travelling 36 mph [57.9 kph] when it hits another car. V-l skidded 52 ft [15.8 m] at a drag factor of 0.77 just prior to impact. What was the speed of V-l at the start of skid?

Solution:

This problem is solved by using the combined speed formula. The variables:

S_1 = Speed of V-l at impact = 36 mph

d = Pre-impact skid distance = 52 ft

f = Drag factor of truck = 0.77

$$S = \sqrt{S_1^2 + 30 * d * f}$$

$$S = \sqrt{36^2 + 30 * 52 * 0.72}$$

$$S = \sqrt{1296 + 1201} = \sqrt{2497}$$

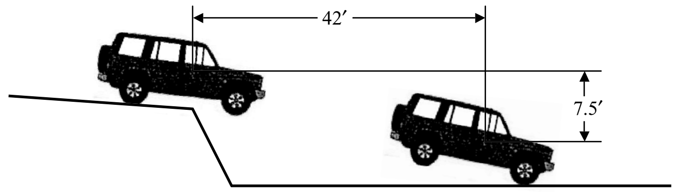
$$S = 49.97 \text{ or } 50 \text{ mph [80 kph]}$$

Problem G.8

a) The acceleration of a car was tested. It was found that the car went 175 ft [53.3 m] in 4.2 s. What was the average acceleration of the car? Approximately how fast was the car going at the end of the test?

b) A motorcycle goes down on its side and slides 186 ft [56.7 m]. The LEVEL friction between the bike and the dry asphalt surface is 0.48. The grade over the slide was measured and found to have an average of 2% downward. Determine the speed of the bike at the start of the slide.

c) A jeep runs off an embankment and goes airborne. The horizontal distance of the jump is 42 ft [12.8 m]. The vertical drop is 7.5 ft [2.3 m]. The takeoff grade is 5% downward. Determine the speed of the jeep at takeoff.



Solution:

a) To calculate the average, we need to utilize the equation that gives the relation between acceleration, initial velocity and distance.

$$d = d_O + V_O*t + 0.5*a*t^2$$

where d = Total distance = 175 ft

d_O = Starting point = 0 ft

V_O = Initial velocity = 0 ft/s

a = acceleration rate

t = time = 4.2 s

Substituting and solving:

$$175 = 0 + 0*4.2 + 0.5*a*4.2^2$$

$$175 = 0.5*a*17.6 = 8.82*a$$

$$a = 175/8.82 = 19.8 \text{ ft/s/s } [6.0 \text{ m/s/s}]$$

Velocity at the end of the test can be estimated by multiplying the acceleration by the time elapsed:

$$V = a*t = 19.8*4.2 = 83.16 \text{ ft/s}$$

Converting to miles per hour:

$$S = V/1.467 = 83.16/1.467 = 56.7 \text{ or } 57 \text{ mph } [96 \text{ kph}]$$

b) To find speed at start of the skid, skid distance, the friction and the basic skid formula. The variables:

S = Speed of the vehicle

d = Skid distance = 186 ft.

m = Slope of the surface = –2% = –2/100 = –0.02 (downward)

f_L = Level friction coefficient = 0.48

f = Drag factor = $f_L + m = 0.48 - 0.02 = 0.46$

Substituting into the formula:

$$S = \sqrt{30 * d * f} = \sqrt{30 * 186 * 0.46}$$

$$S = \sqrt{2567} = 50.67 \text{ or } 51 \text{ mph } [82 \text{ kph}]$$

c) The variables:

S = Speed of vehicle at takeoff

d = Horizontal flight distance = 42 ft

H = Height change = −7.5 ft (downward)

m = Takeoff slope = −5% = −5/100 = −0.05 (downward)

To find speed at takeoff, we use the free fall formula:

$$S = \frac{2.74 * d}{\sqrt{(m * d - H}} = \frac{2.74 * 42}{\sqrt{(-0.05 * 42 - - 7.5)}} = \frac{2.74 * 42}{\sqrt{(-2.1 + 7.5)}}$$

$$S = 115.1/\sqrt{(5.4)} = 115.1/2.32$$

S = 49.53 or 50 mph [80 kph]

Problem G.9

a) A dump truck skids 103 ft [31.4 m] on a dry asphalt surface. The truck/surface level coefficient of friction is 0.45, but the surface has average upslope of 2%. Calculate the speed of the truck at the start of skid.

b) A truck accelerates at 0.14 g's. How much time will it take to go from a standing start to 60 mph [97 kph]? How much distance did it require to go from 0 to 60?

c) A car is traveling at 85 mph [137 kph]. How much distance is required to brake to a stop if the drag factor is 0.74? How much time would be required?

Solution:

a) To find speed at start of the skid, skid distance, the friction and the basic skid formula. The variables:

S = Speed of the vehicle

d = Skid distance = 103 ft.

f = Drag factor = 0.45

m = cross slope = +2% (upward) = +2/100 = +0.02

Substituting into the formula:

$$S = \sqrt{30 * d * (f + m)} = \sqrt{30 * 103 * (.045 + 0.02)} = \sqrt{30 * 103 * 0.47}$$

$$S = \sqrt{1452} = 38.11 \text{ or } 38 \text{ mph } [61 \text{ kph}]$$

b) The variables:

d = Total distance

d_O = Starting point = 0 ft

S = Final speed = 60 mph

V_O = Initial velocity = 0 ft/s

V = Final velocity (in ft/s)

g = acceleration of gravity = 32.2 ft/s/s

a = acceleration rate = 0.14*g = 4.5 ft/s/s

t = time

Converting to miles per hour to feet per second:

$$V = S*1.467 = 60*1.467 = 88.0 \text{ ft/s}$$

Time required can be found by dividing velocity V by acceleration:

$$t = V/a = 88.0/4.5 = 19.5 \text{ s}$$

To calculate the distance required, we need to utilize the equation that gives the relation between acceleration, initial velocity and distance.

$$d = d_O + V_O*t + 0.5*a*t^2$$

$$d = 0 + 0*19.5 + 0.5*4.5*19.5^2$$

$$d = 0.5*4.5*381 = 856 \text{ ft } [261 \text{ m}]$$

c) To find the total distance needed to stop, we use a variant of basic skid formula. The variables:

S = Speed of the car = 85 mph

d = Distance required to brake to stop

f = Drag factor = 0.74

Substituting into the equation:

$$d = S^2/(30*f) = 85^2/(30*0.74)$$

$$d = 7225/22.2$$

$$d = 325.4 \text{ or } 325 \text{ feet } [99 \text{ m}]$$

Problem G.10

A motorcycle goes down on its side and slides 41 ft [12.5 m]. The LEVEL friction between the bike and the dry asphalt surface is 0.52. The grade over the slide was measured and found to have an average of 2% downward. Determine the speed of the bike at the start of the slide.

Solution:

To find speed at start of the skid, skid distance, the friction and the basic skid formula. The variables:

S = Speed of the vehicle d = Skid distance = 41 ft

m = Slope of the surface = –2% = –2 / 100 = –0.02 (downward)

f_L = Level friction coefficient = 0.52

f = Drag factor = f_L + m = 0.52 + –0.02 = 0.50

Substituting into the formula:

$$S = \sqrt{30 * d * f} = \sqrt{30 * 41 * 0.50}$$

$$S = \sqrt{615} = 24.8 \text{ or } 25 \text{ mph } [40 \text{ kph}]$$

Problem G.11

A motorcycle skids 75 ft [23.0 m] with only the rear wheel skidding (drag factor = 0.35). It then skids 28 ft [8.5 m] with both wheels sliding (drag factor = 0.88). It then slides 103 ft [31.4 m] on its side [drag factor = 0.52). Detente the speed of the bike at the start of the one-wheel skid.

Solution:

We have a combined speed problem on our hands. The variables:

S = Speed of motorcycle at start of precrash skid

d_1 = One-wheel skid distance = 75 ft

f_1 = One-wheel skid drag factor = 0.35

d_2 = Two-wheel skid distance = 28 ft

f_2 = Two-wheel skid drag factor = 0.88

d_s = Side slide distance = 103 ft

f_s = Side slide drag factor = 0.52

The speed of V-l at the start of precrash braking:

$$S = \sqrt{30 * d_1 * f_1 + 30 * d_2 * f_2 + 30 * d_s * f_s}$$

$$S = \sqrt{30 * 75 * 0.35 + 30 * 28 * 0.88 + 30 * 103 * 0.52}$$

$$S = \sqrt{787.5 + 739.2 + 1606.8} = \sqrt{3133.5}$$

$$S = 55.97 \text{ or } 56 \text{ mph } [90 \text{ km/h}]$$

Problem G.12

4600-lb [2087 kg] V-2 is stopped at a traffic signal when it is struck in the rear by 3800-lb [1724 kg] eastbound, V-1. After impact, V-l traveled east 21 ft [6.4 m] at an average drag factor of 0.40. After impact V-2 traveled east 28 ft [8.5 m] at an average drag factor of 0.35. Calculate the impact speed of V-l.

Solution:

This is a simple application of the in-line (one dimensional) conservation of momentum formula. The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 0 mph

s_1 = Speed of V-l after impact

s_2 = Speed of V-2 after impact

d_1 = V-l post impact skid distance = 21 ft

d_2 = v-2 post impact skid distance = 28 ft

f_1 = Drag factor of V-l = 0.40

f_2 = Drag factor of V-2 = 0.35

W_1 = Weight of V-l = 3800 1b

W_2 = Weight of V-2 = 4600 lb

The weight ratio: If $W_1 = 1$, $W_2 = 4600/3800 = 1.211$

To find speeds after impact, we use the post impact skid distance and the basic skid formula. For Vehicle One:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 21 * 0.40} = 15.9\,\text{mph}$$

Speed after impact of Vehicle Two:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 28 * 0.35} = 17.1\,\text{mph}$$

The conservation of momentum formula:

$$S_1{*}W_1 + S_2{*}W_2 = s_1{*}W_1 + s_2{*}W_2$$

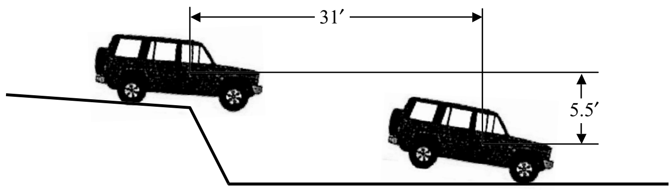
$$S_1{*}1 + 0{*}1.211 = 15.9{*}1 + 17.1{*}1.211$$

$$S_1 = 15.9 + 20.7$$

$$S_1 = 36.6 \text{ or } 37 \text{ mph [59 kph]}$$

Problem G.13

A jeep runs off an embankment and goes airborne. The horizontal distance of the jump is 31 ft [9.4 m]. The vertical drop is 5.5 ft [1.7 m]. The takeoff grade is 3% downward. Determine the speed of the jeep at takeoff.



Solution:

The variables:

S = Speed of vehicle at takeoff

d = Horizontal flight distance = 31 ft

H = Height change = –5.5 ft (downward)

m = Takeoff slope = –3% = –3/100 = –0.03 (downward)

To find speed at takeoff, we use the free fall formula:

$$S = \frac{2.74 * d}{\sqrt{(m * d - H)}} = \frac{2.74 * 31}{\sqrt{(-0.03 * 31 - (-5.5))}} \frac{2.74 * 31}{\sqrt{(-0.93 + 5.5)}}$$

$$S = 84.9/\sqrt{(4.57)} = 84.9/2.14$$

$$S = 39.71 \text{ or } 40 \text{ mph } [64 \text{ kph}]$$

Problem G.14

4600-lb [2087 kg] V-2 is stopped at a traffic signal when it is struck in the rear by 2800-lb [1270 kg] eastbound V-l. After impact V-l traveled east 28 ft [8.5 m] at an average drag factor of 0.40. After impact V-2 traveled east 32 ft [9.8 m] at an average drag factor of 0.35. Calculate the impact speed of V-l.

Solution:

This is a simple application of the in-line (one dimensional) conservation of momentum formula. The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 0 mph

s_1 = Speed of V-l after impact

s_2 = Speed of V-2 after impact

d_1 = V-l post impact skid distance = 28 ft

d_2 = V-2 post impact skid distance = 32 ft

f_1 = Drag factor of V-l = 0.40

f_2 = Drag factor of V-2 = 0.35

W_1 = Weight of V-l = 2800 lb

W_2 = Weight of V-2 = 4600 lb

The weight ratio: If $W_1 = 1$, $W_2 = 4600/2800 = 1.643$

To find speeds after impact, we use the post impact skid distance and the basic skid formula. For Vehicle One:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 28 * 0.40} = 18.3\,\text{mph}$$

Speed after impact of Vehicle Two:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 32 * 0.35} = 18.3\,\text{mph}$$

The conservation of momentum formula:

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 + 0 * 1.643 = 18.3 * 1 + 18.3 * 1.643$$

$$S_1 = 18.3 + 30.1$$

$$S = 48.4 \text{ or } 48 \text{ mph } [78 \text{ km/h}]$$

Problem G.15

For the collision in Problem 3, recalculate V-1's impact speed using dissipation of energy. V-1 has an average of 7 in. [18 cm] of crush on its front. V-2 has an average of 16 in. [41 cm] of crush on the rear. Use the following Campbell equations:

S in mph, C_{AVG} in inches:	V-1 front:	$ebs = 1.46 * C_{MAX} + 7$
	V-2 rear:	$ebs = 1.15 * C_{MAX} + 5$
S in kph, C_{AVG} in cm:	V-1 front:	$ebs = 0.93 * C_{MAX} + 11$
	V-2 rear:	$ebs = 0.73 * C_{MAX} + 8$

Solution:

Solution of this problem can be done in two steps. We will first calculate equivalent barrier speed for the damage on each vehicle. We will then use the dissipation of energy equation to solve for V-l's impact speed. Besides the data from the previous, we will use the following variables:

C_{AVG1} = Average crush depth to V-l front = 7 in.

C_{AVG2} = Average crush depth to V-2 rear = 16 in.

ebs_1 = Equivalent barrier speed of V-l

ebs_2 = Equivalent barrier speed of V-2

Equivalent barrier speed of V-l:

$$ebs_1 = 1.46 * C_{AVG1} + 7 = 1.46 * 7 + 7 = 17.2$$

Equivalent barrier speed of V-2:

$$ebs_2 = 1.15{*}C_{AVG2} + 5 = 1.15{*}16 + 5 = 23.4$$

Substituting and solving the dissipation of energy equation:

$$S_1^2{*}W_1 + S_2^2{*}W_2 = s_1^2{*}W_1 + s_2^2{*}W_2 + ebs_1^2{*}W_1 + ebs_2^2{*}W_2$$

$$S_1^2{*}1 + 0^2{*}1.643 = 18.3^2{*}1 + 18.3^2{*}1.643 + 17.2^2{*}1 + 23.4^2{*}1.643$$

$$S_1^2 = 335 + 550 + 296 + 900 = 2081$$

$$S_1 = \sqrt{2081} = 45.6 \text{ or } 46 \text{ mph } [73 \text{ km/h}]$$

Problem G.16

4600-pound [2087 kg] V-2 is stopped at a traffic signal when it is struck in the rear by 3800-pound [1724 kg] eastbound V-l. After impact V-l traveled east 51 feet [15.5 m] at an average drag factor of 0.40. After impact V-2 traveled east 60 feet [18.3 m] at an average drag factor of 0.35. Calculate the impact speed of V-l.

Solution:

This is a simple application of the in-line (one dimensional) conservation of momentum formula. The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 0 mph

s_1 = Speed of V-l after impact

s_2 = Speed of V-2 after impact

d_1 = V-l post impact skid distance = 51 ft

d_2 = V-2 post impact skid distance = 60 ft

f_1 = Drag factor of V-l = 0.40

f_2 = Drag factor of V-2 = 0.35

W_1 = Weight of V-l = 3800 1b

W_2 = Weight of V-2 = 4600 lb

The weight ratio: If $W_1 = 1$, $W_2 = 4600/3800 = 1.211$

To find speeds after impact, we use the post impact skid distance and the basic skid formula. For Vehicle One:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 51 * 0.40} = 24.7 \text{ mph}$$

Speed after impact of Vehicle Two:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 60 * 0.35} = 25.1 \text{ mph}$$

The conservation of momentum formula:

$$S_1{*}W_1 + S_2{*}W_2 = s_1{*}W_1 + s_2{*}W_2$$

$$S_1{*}1 + 0{*}1.211 = 24.7{*}1 + 25.1{*}1.211$$

$$S_1 = 24.7 + 30.4$$

$$S_1 = 55.1 \text{ or } 55 \text{ mph } [88 \text{ kph}]$$

Problem G.17

A pickup truck skids 51 ft [15.5 m] on a dirt surface (drag factor of 0.63), skids another 59 ft [18 m] on a grass surface (drag factor of 0.38) and hits a one foot [0.3 m] diameter tree head-on. The impact leaves a 28-in. [71 cm] deep dent in the front of the pickup. Using one of the equations below,

S in mph, C_{MAX} in inches:	$S = 0.84{*}C_{MAX} + 4.0$
S in kph, C_{MAX} in cm:	$S = 0.53{*}C_{MAX} + 6.4$

Calculate the speed of the pickup at impact. Also calculate the speed of the pickup at start of skid on the dirt.

Solution:

Because there is no significant post impact movement to consider finding impact speed (S_1) is easy. Simply substitute the 28 in. for C_{MAX} in the equation and crunch the numbers:

$$S = 0.84{*}C_{MAX} + 4.0$$

$$S = 0.84{*}28 + 4.0 = 23.52 + 4.0$$

$$S_1 = 27.52 \text{ or } 28 \text{ mph } [44 \text{ kph}]$$

The speed at the start of skid can be solved is by using the combined speed formula. The variables:

S_1 = Speed of V-l at impact = 27.5 mph

d_A = Skid distance on dirt = 51 ft

f_A = Drag factor on dirt = 0.63

d_G = Skid distance on grass = 59 ft

f_G = Drag factor on grass = 0.38

$$S = \sqrt{S_1^2 + 30 * d_D f_D + 30 * d_G + f_G}$$

$$S = \sqrt{27.5^2 + 30 * 51 * 0.63 + 30 * 59 * 0.38}$$

$$S = \sqrt{756 + 964 + 673} = \sqrt{239}$$

$$S = 48.91 \text{ or } 49 \text{ mph } [79 \text{ kph}]$$

Problem G.18

An automobile begins to yaw on a banked curve. The critical speed scuff mark laid down by the outside front tire was measured with a 60-ft [18.3 m] chord and found to have a middle ordinate of 20.5 in. [52 cm]. The LEVEL coefficient of friction is 0.79. Along the scuff mark the surface is banked 1%. Determine the speed of the vehicle.

Solution:

Solution of this problem has two main steps. We will ultimately utilize the critical speed to sideslip equation (banked surface). First, we must determine the radius of curvature of the tire mark used to estimate the radius of curvature of the center of mass. The equation:

$$R = \frac{C^2}{8 * m_O} + \frac{m_O}{2}$$

where R = Radius of curvature, feet

C = Chord length = 60 ft

m_O = Middle ordinate length, feet = 20.5"/12 = 1.71 ft

Substituting and solving:

$$R = \frac{60^2}{8 * 1.71} + \frac{1.71}{2} = 264\,\text{ft}$$

We now plug the radius of curvature, lateral friction coefficient and cross slope into the critical speed to sideslip formula. The equation:

$$S = \frac{\sqrt{15 * R * (f + m)}}{\sqrt{(1 - f * m)}}$$

where S = Speed in miles per hour

f = Lateral friction coefficient = .79

R = Radius of curvature = 264 ft

m = cross slope = 1% = 1/100 = 0.01

Substituting and solving:

$$S = \frac{\sqrt{15 * 264 * (0.79 + 0.01)}}{\sqrt{(1 - 0.79 * 0.01)}} = \frac{\sqrt{3168}}{\sqrt{0.992}}$$

S = 56.28/0.996 = 56.51 or 57 mph [91 km/h]

Problem G.19

A Camaro is drag racing another vehicle when it goes out of control, hits a curb, and goes airborne. As it leaves the ground it scrapes the top off a small dirt mound. The scrape is measured and found to have a takeoff slope of 8%. The horizontal distance of the jump is 109 ft [33.2 m]. There is no change in elevation from take-off to landing. Determine the take-off speed of the Camaro.

Solution:

The variables:

S = Speed of vehicle at takeoff

d = Horizontal flight distance = 109 ft

H = Height change = 0 ft

m = Takeoff slope = +8% = +8/100 = +0.08 (upward)

To find speed at takeoff, we use the free fall formula:

$$S = \frac{2.74 * d}{\sqrt{(m * d - H)}} = \frac{2.74 * 109}{\sqrt{(0.08 * 109 - 0)}} = \frac{2.74 * 109}{\sqrt{(8.7 - 0)}}$$

$$S = 298.7/\sqrt{(8.7)} = 298.7/2.95$$

$S = 101.15$ or 101 mph [163 kph]

Problem G.20

For the collision in Problem 3, recalculate V-l's impact speed using dissipation of energy. V-l has an average of 4 in. [10 cm] of crush on its front. V-2 has an average of 12 in. [30 cm] of crush on the rear. Use the following Campbell equations:

S in mph, C_{AVG} in inches:	V-l front:	$ebs = 1.46*C_{MAX} + 7$
	V-2 rear:	$ebs = 1.15*C_{MAX} + 5$
S in kph, C_{AVG} in cm:	V-l front:	$ebs = 0.93*C_{MAX} + 11$
	V-2 rear:	$ebs = 0.73*C_{MAX} + 8$

Solution:

Solution of this problem can be done in two steps. We will first calculate equivalent barrier speed for the damage on each vehicle. We will then use the dissipation of energy equation to solve for V-l's impact speed. Besides the data from the previous, we will use the following variables:

C_{AVG1} = Average crush depth to V-l front = 4 in.

C_{AVG2} = Average crush depth to V-2 rear = 12 in.

ebs_1 = Equivalent barrier speed of V-l

ebs_2 = Equivalent barrier speed of V-2

Equivalent barrier speed of V-1:

$$ebs_1 = 1.46*C_{AVG1} + 7 = 1.46*4 + 7 = 12.8$$

Equivalent barrier speed of V-2:

$$ebs_2 = 1.15 * C_{AVG2} + 5 = 1.15*12 + 5 = 18.8$$

Substituting and solving the dissipation of energy equation:

$$S_1^2*W_1 + S_2^2*W_2 = s_1^2*W_1 + s_2^2*W_2 + ebs_1^2*W_1 + ebs_2^2*W_2$$

$$S_1^2*1 + 0^2*1.211 = 15.9^2*1 + 17.1^2*1.211 + 12.8^2*1 + 18.8^2*1.211$$

$$S_1^2 = 253 + 354 + 164 + 428 = 1199$$

$$S_1 = \sqrt{1199} = 34.6 \text{ or } 35 \text{ mph [56 kph]}$$

Problem G.21

For the collision in Problem 3, recalculate V-1's impact speed using dissipation of energy. V-1 has an average of 10 in. [25 cm] of crush on its front. V-2 has an average of 27 in. [69 cm] of crush on the rear. Use the following Campbell equations:

S in mph, C_{AVG} in inches:	V-1 front:	$ebs = 1.46*C_{MAX} + 7$
	V-2 rear:	$ebs = 1.15*C_{MAX} + 5$
S in kph, C_{AVG} in cm:	V-1 front:	$ebs = 0.93*C_{MAX} +11$
	V-2 rear:	$ebs = 0.73*C_{MAX} + 8$

Solution:

Solution of this problem can be done in two steps. We will first calculate equivalent barrier speed for the damage on each vehicle. We will then use the dissipation of energy equation to solve for V-1's impact speed. Besides the data from the previous, we will use the following variables:

C_{AVG1} = Average crush depth to V-1 front = 10 in.

C_{AVG2} = Average crush depth to V-2 rear = 27 in.

ebs_1 = Equivalent barrier speed of V-1

ebs_2 = Equivalent barrier speed of V-2

Equivalent barrier speed of V-1:

$$ebs_1 = 1.46*C_{AVG1} + 7 = 1.46*10 + 7 = 21.6$$

Equivalent barrier speed of V-2:

$$ebs_2 = 1.15 * C_{AVG2} + 5 = 1.15*24 + 5 = 36.0$$

Substituting and solving the dissipation of energy equation:

$$S_1^2{*}W_1 + S_2^2{*}W_2 = s_1^2{*}W_1 + s_2^2{*}W_2 + ebs_1^2{*}W_1 + ebs_2^2{*}W_2$$

$$S_1^2{*}1 + 0^2{*}1.211 = 24.7^2{*}1 + 25.1^2{*}1.211 + 21.6^2{*}1 + 36.0^2{*}1.211$$

$$S_1^2 = 610 + 739 + 467 + 1569 = 3385$$

$$S_1 = \sqrt{3385} = 28.2 \text{ or } 58 \text{ mph [94 kph]}$$

Problem G.22

A pickup truck skids 50 ft [15.2 m] on a dirt surface (drag factor of 0 63), skids another 62 ft [18.9 m] on a grass surface (drag factor of 0.42) and hits a 1-ft [0.3 m] diameter tree head-on. The impact leaves a 27-in. [68.5 cm] deep dent in the front of the pickup. Using one of the equations below,

S in mph, C_{MAX} in inches: $S = 0.84{*}C_{MAX} + 4.0$

S in kph, C_{MAX} in cm: $S = 0.53{*}C_{MAX} + 6.4$

calculate the speed of the pickup at impact. Also calculate the speed of the pickup at start of skid on the dirt.

Solution:

Because there is no significant post impact movement to consider, finding impact speed (S_1) is easy. Simply substitute the 27 in. for C_{MAX} in the equation and crunch the numbers:

$$S = 0.84{*}C_{MAX} + 4.0$$

$$S = 0.84{*}27 + 4.0 = 22.7 + 4.0$$

$$S = 26.7 \text{ or } 27 \text{ mph [43 kph]}$$

The speed at the start of skid can be solved is by using the combined speed formula. The variables:

S = Speed of V-l at impact = 26.7 mph

d_A = Skid distance on dirt = 50 ft

f_A = Drag factor on dirt = 0.63

d_G = Skid distance on grass = 62 ft

f_G = Drag factor on grass = 0.42

$$S = \sqrt{S_1^2 + 30 * d_D * f_D + 30 * d_G * f_G}$$

$$S = \sqrt{26.7^2 + 30 * 50 * 0.63 + 30 * 62 * 0.42}$$

$$S = \sqrt{713 + 945 + 781} = \sqrt{2439}$$

$$S = 49.45 \text{ or } 49 \text{ mph [79 kph]}$$

Problem G.23

A 53,800-lb [24,400 kg] tractor-trailer is traveling 47 mph [75 kph] when it is struck in the rear by a 3450-lb [1565 kg] auto. The car's EDR shows the vehicle was traveling at a constant 82 mph [132 kph] before impact. If the vehicles remain together immediately after impact, what are the Delta V's from impact for both vehicles?

Solution:

This is an unusual application of the in-line (one dimensional) conservation of momentum formula. We are solving for POST impact speed. We then use that to calculate delta Vs. The variables:

S = Speed of V-l at impact = 82 mph

S_2 = Speed of V-2 at impact = 47 mph

s = Speed of V-l and V-2 after impact

W_1 = Weight of V-l = 3450 1b

W_2 = Weight of V-2 = 53,800 1b

The weight ratio: If $W_1 = 1$, $W_2 = 53800/3450 = 15.59$

The conservation of momentum formula:

$$S_1{*}W_1 + S_2{*}W_2 = s{*}W_1 + s{*}W_2$$

$$82{*}1 + 47{*}15.59 = s{*}l + s{*}15.59$$

$$82 + 732 = s{*}16.59$$

$$s = 49.1 \text{ mph}$$

Delta V of Auto: $s - S_1 = 49.1 - 82 = -32.9$ mph [–53 kph]

Delta V of Truck: $s - S_2 = 49.1 - 47 = 2.1$ mph [3 kph]

Problem G.24

According to the NHTSA NCAP crash test, a 2005 Chevrolet Uplander struck a full-length rigid barrier at 34.9 mph [56.2 km/h] and ended up with 22.8 in. [58 cm] of crush (average) across the front. Using a threshold of measureable damage of 7 mph [11 km/h], determine the Campbell equation for the front of this vehicle. Calculate the 'A,' 'B,' and 'G' stiffness coefficients that would be used in the CRASH computer program for the truck's front. Use a damage width of 70 in. [178 cm] and a curb weight of 4392 lb [1992 kg].

Solution:

The Campbell model, in equation form:

$$ebs = b_1{*}C_{AVG} + b_0$$

where ebs = Equivalent barrier speed, mph = 34.9 mph

b_1 = Stiffness constant, mph/in

C_{AVG} = Average crush depth, in = 27.8

b_0 = Damage threshold constant, given 7 mph in this problem

The equation can also be generated for metric units. The b_1 and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph $\quad ebs = b_1 * C_{AVG} + b_0$

$$34.9 = b_1 * 22.8 + 7$$

$$27.9 = b_1 * 22.8$$

$$1.22 = b_1$$

Therefore, the equation:

$$ebs = 1.22 * C_{AVG} + 7$$

The CRASH3 stiffness coefficients can also be derived for the vehicle. We will use curb weight (W = 4392 lb), the average crush, a 7 mph damage threshold, and the damage width (L = 70 in.).

3. Because we are using 7 mph as the damage threshold constant, the b_1 and b_0 constants will be the same, except that mph must be converted Co inches per second:

$$b_0 = 7 \text{ mph} * \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in.}}{\text{ft}} = 123.2 \text{ in./s}$$

$$b_1 = \frac{1.22 \text{ mph}}{\text{in.}} * \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in.}}{\text{ft}} = 21.5/\text{s}$$

The CRASH3 constants:

4. $$A = \frac{W * b_0 * b_1}{g * L} = \frac{4392 * 123.2 * 21.5}{382.4 * 70} = 435$$

$$B = \frac{W * b_1 * b_1}{g * L} = \frac{4392 * 21.5 * 21.5}{382 * 70} = 76$$

$$G = A^2/2*B = 435^2/2*76 = 1247$$

Note that 'g,' the acceleration of gravity, is given as (32.2*12) inches per second per second in order to make the units consistent.

Problem G.25

2700-lb [1225 kg] V-2 is stopped at a traffic signal when it is struck in the rear by 3800-lb [1724 kg] eastbound V-l. After impact, V-l traveled east 18 ft [5.5 m] at an average drag factor of 0.40. After impact V-2 traveled east 22 ft [6.7 m] at an average drag factor of 0.35. Calculate the impact speed of V-l.

Solution:

This is a simple application of the in-line (one dimensional) conservation of momentum formula. The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 0 mph

s_1 = Speed of V-l after impact

s_2 = Speed of V-2 after impact

d_1 = V-l post impact skid distance = 18 ft

d_2 = V-2 post impact skid distance = 22 ft

f_1 = Drag factor of V-l = 0.40

f_2 = Drag factor of V-2 = 0.35

W_1 = Weight of V-l = 3800 1b

W_2 = Weight of V-2 = 2700 lb

The weight ratio: If $W_1 = 1$, $W_2 = 2700/3800 = 0.710$

To find speeds after impact, we use the post impact skid distance and the basic skid formula. For Vehicle One:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 18 * 0.40} = 14.7 \text{ mph}$$

Speed after impact of Vehicle Two:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 22 * 0.35} = 15.2 \text{ mph}$$

The conservation of momentum formula:

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 + 0 * 0.710 = 14.7 * 1 + 15.2 * 0.710$$

$$S_1 = 14.7 + 10.8$$

$$S_1 = 25.5 \text{ or } 25 \text{ mph [41 kph]}$$

Problem G.26

An automobile begins to yaw on a banked curve. The critical speed scuff mark laid down by the outside front tire was measured with a 60 ft [18.3 m] chord and found to have a middle ordinate of 17.5 in. [44 cm]. The LEVEL coefficient of friction is 0.81. Along the scuff mark the surface is banked 4%. Determine the speed of the vehicle.

Solution:

Solution of this problem has two main steps. We will ultimately utilize the critical speed to sideslip equation (banked surface). First we must determine the radius of curvature of the tire mark used to estimate the radius of curvature of the center of mass. The equation:

$$R = \frac{C^2}{8 * m_O} + \frac{m_O}{2}$$

where R = Radius of curvature, feet

C = Chord length = 60 ft

m_O = Middle ordinate length, feet = 17.5"/12 = 1.46 ft

Substituting and solving:

$$R = \frac{60^2}{8 * 1.46} = \frac{1.46}{2} = 310 \text{ ft}$$

We now plug the radius of curvature, lateral friction coefficient and cross slope into the critical speed to sideslip formula. The equation:

$$S = \frac{\sqrt{15 * R * (f + m)}}{\sqrt{(1 - f * m)}}$$

where S = Speed in miles per hour

f = Lateral friction coefficient = .81

R = Radius of curvature = 310 ft

m = cross slope = 4% = 4/100 = 0.04

Substituting and solving:

$$S = \frac{\sqrt{15 * 310 * (0.81 + 0.04)}}{\sqrt{(1 - 0.81 * 0.04)}} = \frac{\sqrt{3952}}{\sqrt{0.968}}$$

$$S = 62.87/0.984 = 63.91 \text{ or } 64 \text{ mph } [103 \text{ kph}]$$

Problem G.27

4000-lb [1814 kg] V-2 is stopped at a traffic signal when it is struck in the rear by 3800-lb [1724 kg] eastbound V-l. After impact, V-l traveled east 31 ft [9.4 m] at an average drag factor of 0.40. After impact V-2 traveled east 42 ft [12.8 m] at an average drag factor of 0.3 5. Calculate the impact speed of V-l.

Solution:

This is a simple application of the in-line (one dimensional) conservation of momentum formula. The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 0 mph

s_1 = Speed of V-l after impact

s_2 = Speed of V-2 after impact

d_1 = V-l post impact skid distance = 31 ft

d_2 = V-2 post impact skid distance = 42 ft

f_1 = Drag factor of V-l = 0.40

f_2 = Drag factor of V-2 = 0.35

W_1 = Weight of V-l = 3800 1b

W_2 = Weight of V-2 = 4000 lb

The weight ratio: If $W_1 = 1$, $W_2 = 4000/3800 = 1.053$

To find speeds after impact, we use the post impact skid distance and the basic skid formula. For Vehicle One:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 21 * 0.40} = 19.3\,\text{mph}$$

Speed after impact of Vehicle Two:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 42 * 0.35} = 21.0\,\text{mph}$$

The conservation of momentum formula:

$$S_1{*}W_1 + S_2{*}W_1 = s_1{*}W_1 + s_2{*}W_2$$

$$S_1{*}1 + 0{*}1.053 = 19.3{*}1 + 21.0{*}1.053$$

$$S_1 = 19.3 + 22.1$$

$$S_1 = 41.4 \text{ or } 41 \text{ mph [67 kph]}$$

Problem G.28

An automobile begins to yaw on a banked curve. The critical speed scuff mark laid down by the outside front tire was measured with a 60 foot [18.3 m] chord and found to have a middle ordinate of 18.5 in. [47 cm]. The LEVEL coefficient of friction is 0.76. Along the scuff mark the surface is banked 2%. Determine the speed of the vehicle.

Solution:

Solution of this problem has two main steps. We will ultimately utilize the critical speed to sideslip equation (banked surface). First, we must determine the radius of curvature of the tire mark used to estimate the radius of curvature of the center of mass. The equation:

$$R = \frac{C^2}{8 * m_O} + \frac{m_O}{2}$$

where R = Radius of curvature, feet

C = Chord length = 60 ft

m_O = Middle ordinate length, feet = 18.5"/12 = 1.54 ft

Substituting and solving:

$$R = \frac{60^2}{8 * 1.54} + \frac{1.54}{2} = 293\,\text{ft}$$

We now plug the radius of curvature, lateral friction coefficient and cross slope into the critical speed to sideslip formula. The equation:

$$S = \frac{\sqrt{15 * R * (f + m)}}{\sqrt{(1 - f * m)}}$$

where: S = Speed in miles per hour

f = Lateral friction coefficient = .76

R = Radius of curvature = 293 ft

m = cross slope = 2% = 2/100 = 0.02

Substituting and solving:

$$S = \sqrt{\frac{15 * 293 * (0.76 + 0.2)}{\sqrt{(1 - 0.76 * 0.2)}}} = \frac{\sqrt{3428}}{\sqrt{0.985}}$$

S = 58.55/0.985 = 58.96 or 59 mph [95 kph]

Problem G.29

a) A 46,800-lb [21,200 kg] tractor-trailer is traveling 55 mph [88 kph] when it is struck in the rear by a 3450-lb [1565 kg] auto. The car's EDR shows the vehicle was traveling at a constant 93 mph [150 kph] before impact. If the vehicles remain together immediately after impact, what were the Delta V's from impact for both vehicles?

b) In Problem (a), if the driver of the car began hard braking (f = 0.75) 2 s before impact, would the collision had been avoided?

Solution:

a) This is an unusual application of the in-line (one dimensional) conservation of momentum formula. We are solving for POST impact speed. We then use that to calculate delta V's. The variables:

S_1 = Speed of V-l at impact = 93 mph

S_2 = Speed of V-2 at impact = 55 mph

s = Speed of V-1 and V-2 after impact

W_1 = Weight of V-l = 3450 lb

W_2 = Weight of V-2 = 46,800 lb

The weight ratio: If $W_1 = 1$, $W_2 = 46800/3450 = 13.56$

The conservation of momentum formula:

$$S_1{*}W_1 + S_2{*}W_2 = s{*}W_1 + s{*}W_2$$

$$93{*}1 + 55{*}13.56 = s{*}1 + s{*}13.56$$

$$93 + 746 = s{*}14.56$$

$$s = 51.2 \text{ mph}$$

Delta V of Auto: $S_{1-s} = 93 - 51.2 = 41.8$ mph [6.1 kph]

Delta V of Truck: $S_{2-s} = 55 - 51.2 = 3.8$ mph [6.1 kph]

b) If we can figure out how much speed would be lost in two seconds of hard braking, we could subtract that speed from the car's initial and see if it reduces speed below the truck's speed. The deceleration, a, from braking at f = 0.75:

$$a = f*g = 0.75*32.2 = 24.15 \text{ ft/s/s}$$

Velocity loss in 2 s:

$$V = a*t = 24.15*2 = 48.3 \text{ ft/sc}$$

Converting to speed S:

$$S = V/1.467 = 48.3/1.467 = 32.9 \text{ mph}$$

The car's speed would be reduced to:

$$93 - 32.9 = 60.1 \text{ mph}$$

The collision would still have occurred, but the impact would have been minor.

Problem G.30

An automobile begins to yaw on a banked curve. The critical speed scuff mark laid down by the outside front tire was measured with a 60-ft [18.3 m] chord and found to have a middle ordinate of 15.5 in. [39.4 cm]. The LEVEL coefficient of friction is 0.73. Along the scuff mark, the surface is banked 2%. Determine the speed of the vehicle.

Solution:

Solution of this problem has two main steps. We will ultimately utilize the critical speed to sideslip equation (banked surface). First, we must determine the radius of curvature of the tire mark used to estimate the radius of curvature of the center of mass. The equation:

$$R = \frac{C^2}{8 * m_O} + \frac{m_O}{2}$$

where R = Radius of curvature, ft

C = Chord length = 60 ft

M_O = Middle ordinate length, feet = 15.5"/12 = 1.29 ft

Substituting and solving:

$$R = \frac{60^2}{8 - 1.29} + \frac{1.29}{2} = 348 \text{ ft}$$

We now plug the radius of curvature, lateral friction coefficient and cross slope into the critical speed to sideslip formula. The equation:

$$S = \frac{\sqrt{15 * R * (f + m)}}{\sqrt{(1 - f * m)}}$$

where S = Speed in miles per hour

f = Lateral friction coefficient = 0.73

R = Radius of curvature = 348 ft

m = cross slope = 2% = 2/100 = 0.02

Substituting and solving:

$$S = \frac{\sqrt{15 * 348 * (0.73 + 0.02)}}{\sqrt{(1 - 0.73 * 0.02)}} = \frac{\sqrt{3919}}{\sqrt{0.985}}$$

S = 62.60/0.993 = 63.04 or 63 mph [101 kph]

Problem G.31

An automobile begins to yaw on a banked curve. The critical speed scuff mark laid down by the outside front tire was measured with a 60-ft [18.3 m] chord and found to have a middle ordinate of 16.5 in. [42 cm]. The LEVEL coefficient of friction is 0.78. Along the scuff mark the surface is banked 2%. Determine the speed of the vehicle.

Solution:

Solution of this problem has two main steps. We will ultimately utilize the critical speed to sideslip equation (banked surface). First, we must determine the radius of curvature of the tire mark used to estimate the radius of curvature of the center of mass. The equation:

$$R = \frac{C^2}{8 * m_O} + \frac{m_O}{2}$$

where R = Radius of curvature, feet

C = Chord length = 60 ft

m_O = Middle ordinate length, ft = 16.5"/12 = 1.375 ft

Substituting and solving:

$$R = \frac{60^2}{s * 1.375} + \frac{1.375}{2} = 328\,\text{ft}$$

We now plug the radius of curvature, lateral friction coefficient and cross slope into the critical speed to sideslip formula. The equation:

$$S = \frac{\sqrt{15 * R * (f + m)}}{\sqrt{(1 - f * m)}}$$

where S = Speed in miles per hour

f = Lateral friction coefficient = 0.78

R = Radius of curvature = 328 ft

m = cross slope = 2% = 2/100 = 0.02

Substituting and solving:

$$S = \frac{\sqrt{15 * 328 * (0.78 + 0.2)}}{\sqrt{(1 - 0.78 * 0.02}} = \frac{\sqrt{3936}}{\sqrt{0.984}}$$

S = 62.74/0.991 = 63.24 or 63 mph [102 kph]

Problem G.32

According to the NHTSA NCAP crash test, a 2004 Kia Spectra struck a full length rigid barrier at 34.9 mph [56.2 km/h] and ended up with 18.1 in. [46 cm] of crush (average) across the front. Using a threshold of measureable damage of 7 mph [11 km/h], determine the Campbell equation for the front of this vehicle. Calculate the 'A,' 'B,' and 'G' stiffness coefficients that would be used in the CRASH computer program for the truck's front. Use a damage width of 64.1 in. [163 cm] and a curb weight of 2728 lb [1237 kg].

Solution:

The equation can also be generated for metric units. The b_1 and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph

$$ebs = b_1 * C_{AVG} + b_0$$

$$34.9 = b_1 * 18.1 + 7$$

$$27.9 = b_1 * 18.1$$

$$1.54 = b_1$$

Therefore, the equation:

$$ebs = 1.54 * C_{AVG} + 7$$

The CRASH3 stiffness coefficients can also be derived for the Kia. We will use curb weight (W = 2728 lb), the average crush, a 7-mph damage threshold and the damage width (L = 64.1 in.).

Because we are using 7 mph as the damage threshold constant, the b_1 and b_0 constants will be the same, except that mph must be converted to inches per second:

$$b_0 = 7 \text{ mph} * \frac{1.467 \text{ ft/s}}{\text{mph}} = \frac{12 \text{ in}}{\text{ft}} = 123.2 \text{ in./s}$$

$$b_1 = \frac{1.54 \text{ mph}}{\text{in}} * \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in}}{\text{ft}} = 27.1/\text{s}$$

The CRASH3 constants:

$$A = \frac{W * b_0 * b_1}{g * L} = \frac{2728 * 123.2 * 27.1}{382.4 * 64.1} = 372$$

$$B = \frac{W * b_1 - b_1}{g * L} = \frac{2728 * 27.1 * 27.1}{382.4 * 66.4} = 82$$

$$G = A^2/2*B = 372^2/2*82 = 847$$

Note that 'g,' the acceleration of gravity, is given as (32.2*12) inches per second per second in order to make the units consistent.

Problem G.33

According to the NHTSA NCAP crash test, a 2003 Nissan 350Z struck a full length rigid barrier at 34.6 mph [55.7 km/h] and ended up with 16.7 in. [42.5 cm] of crush (average) across the front. Using a threshold of measureable damage of 7 mph [11 km/h], determine the Campbell equation for the front of this vehicle. Calculate the 'A,' 'B,' and 'G' stiffness coefficients that would be used in the CRASH computer program for the track's front. Use a damage width of 66.4 in. [167 cm] and a curb weight of 3329 lb [1510 kg].

Solution:

The Campbell mode!, in equation form:

$$ebs = b_1 * C_{AVE} + b_0$$

where ebs = Equivalent barrier speed, mph = 34.6 mph

b_1 = Stiffness constant, mph/in.

C_{AVG} = Average crush depth, in. = 16.7"

b_0 = Damage threshold constant, given 7 mph in this problem

The equation can also be generated for metric units. The b_1 and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph

$$ebs = b_1 * C_{AVG} + b_0$$

$$34.6 = b_1 * 16.7 + 7$$

$$27.6 = b_1 * 16.7$$

$$1.65 = b_1$$

Therefore, the equation:

$$ebs = 1.65 * C_{AVG} + 7$$

The CRASH3 stiffness coefficients can also be derived for the Nissan. We will use curb weight (W = 3329 lb), the average crush, a 7 mph damage threshold, and the damage width (L = 66.4 in.).

Because we are using 7 mph as the damage threshold constant, the b_1 and b_0 constants will be the same, except that mph must be converted to inches per second:

$$b_0 = 7 \text{ mph} + \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in}}{\text{ft}} = 123.2 \text{ in./s.}$$

$$b_1 = \frac{1.65 \text{ mph}}{\text{in}} * \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in}}{\text{ft}} = 29.0 \text{ s.}$$

The CRASH3 constants:

$$A = \frac{W * b_0 * b_1}{g * L} = \frac{3329 * 123.2 * 29.0}{382.4 * 66.4} = 468$$

$$B = \frac{W * b_1 * b_1}{g * L} = \frac{3329 * 29.0 * 29.0}{382.4 * 66.4} = 110$$

$$G = A^2/2*B = 468^2/2*110 = 993$$

Note that ‘g,’ the acceleration of gravity, is given as (32.2*12) inches per second per second in order to make the units constant.

Problem G.34

A Nissan (weight: 3426 lb [1554 kg]) crosses the centerline and crashes head-on into a late model Ford (weight: 4075 lb [1848 kg]). Neither vehicle moves significantly after impact. Data from the Ford’s EDR was downloaded and the Ford was found to have a longitudinal delta V of 33.7 mph [54.2 kph]. Assuming the delta V reading is accurate, determine the impact speed of the Nissan.

Solution:

Because the Ford was hit head-on and stopped dead in its tracks, its impact speed equals its delta V. Now the problem is a simple straight line momentum problem We will consider the Ford’s speed negative because it is in the opposite direction of the Nissan.

The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 33.7 mph

s = Speed of vehicles after impact = 0 mph

W_1 = Weight of V-l = 3426 lb

W_2 = Weight of V-2 = 4075 lb

The weight ratios: If $W_1 = 1$, then $W_2 = 4075/3426 = 1.189$

The conservation of momentum formula (S, is negative because V-2 is moving in the opposite direction as V-l):

$$S_1*W_1 + S_2*W_2 = s_1*W + s*W_2$$

$$S_1*1 + -33.7*1.189 = 0*1 + 0*1.189 = 0$$

$$S_1 = 33.7*1.189 = 40.08 \text{ or } 40 \text{ mph } [64 \text{ kph}]$$

Problem G.35

A 2005 Ford Mustang, driven by an intoxicated driver, is approaching a ‘T’ intersection. The driver fails to notice the stop sign in time. The truck brakes hard for 53 ft [16.2 m] on the asphalt road surface, 8 ft [2.4 m] grassy dirt surface, and collides head-on with a concrete retaining wall. The front of the truck is later measured and found to have an average crush of 22.5 in. [57 cm]. The drag factors were 0.80 for the asphalt surface and 0.48 for the dirt/grass surface. Estimate both the truck’s impact speed and its speed at the start of precrash skid. The Mustang was tested at 35.1 and suffered 16.8 in. of crush.

Solution:

We will derive the Campbell equation and then use it to estimate the impact speed into the brick wall. We will then use the combined speed formula to estimate speed at the start of skid. The Mustang was crash tested at 35.1 and suffered 16.8 in. of crush.

The Campbell model, in equation form:

$$ebs = b_1 * C_{AVG} + b_0$$

where ebs = Equivalent barrier speed = 35.1 mph

b_1 = Stiffness constant, mph/in

C_{AVG} = Average crush depth = 16.8 in.

b_0 = Damage threshold constant, given 7 mph in this problem

The equation can also be generated for metric units. The b_1 and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph

$$ebs = b_1 * C_{AVG} + b_0$$

$$35.1 = b_1 * 16.8 + 7$$

$$28.1 = b_1 * 16.8$$

$$1.67 = b.$$

Therefore, the equation:

$$ebs = 1.67 * C_{AVG} + 7$$

Because the car hit a fixed object and stopped dead, ebs equals impact speed:

$$S_1 = 1.67 * C_{AVG} + 7 = 1.67 * 22.5 + 7 = 44.6 \text{ mph}$$

The speed at the start of skid can be solved is by using the combined speed formula. The variables:

S_1 = Speed of V-l at impact = 44.6 mph

d_A = Skid distance on asphalt = 53 ft

f_A = Drag factor on asphalt = 0.80

d_G = Skid distance on grass/dirt = 8 ft

f_G = Drag factor on grass/dirt = 0.48

$$S = \sqrt{S_1^2 + 30 * d_A * f_A + 30 * d_G * f_G}$$

$$S = \sqrt{44.6^2 + 30 * 53 * 0.80 + 30 * 8 * 0.48}$$

$$S = \sqrt{1989 + 1272 + 115} = \sqrt{3376}$$

58.1 or 58 mph [94 km/h]

Problem G.36

A 2004 Ford F150 Supercab, driven by an intoxicated driver, is approaching a T intersection. The driver fails to notice the stop sign in time. The truck brakes hard for

43 ft [13.1 m] on the asphalt road surface, 8 ft [2.4 m] grassy dirt surface, and collides head-on with a concrete retaining wall. The front of the truck is later measured and found to have an average crush of 22.5 in. [57 cm]. The drag factors were 0.80 for the asphalt surface and 0.48 for the dirt/grass surface. Estimate both the truck's impact speed and its speed at the start of precrash skid. (Hint: See test data on this vehicle.)

Solution:

We will derive the Campbell equation and then use it to estimate the impact speed into the brick wall. We will then use the combined speed formula to estimate speed at the start of skid. From previous data, we learn that the F150 Supercab was crash tested at 35.1 and suffered 21.7 in. of crush.

The Campbell model, in equation form:

$$ebs = b_1 + C_{AVG} + b_0$$

where ebs = Equivalent barrier speed = 35.1 mph

b_l = Stiffness constant, mph/in

C_{AVG} = Average crush depth = 21.7 in.

b_0 = Damage threshold constant, given 7 mph in this problem

The equation can also be generated for metric units. The b_1 and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph

$$ebs = b_1 {*} C_{AVG} + b_0$$

$$35.1 = b_1 {*} 21.7 + 7$$

$$28.1 = b_1 {*} 21.7$$

$$1.29 = b.$$

Therefore, the equation:

$$ebs = 1.29 {*} C_{AVG} + 7$$

Because the car hit a fixed object and stopped dead, ebs equals impact speed:

$$S1 = 1.29 {*} C_{AVG} + 7 = 1.29 {*} 22.5 + 7 = 36.0 \text{ mph}$$

The speed at the start of skid can be solved is by using the combined speed formula. The variables:

S_1 = Speed of V-l at impact = 36.0 mph

d_A = Skid distance on asphalt = 43 ft

f_A = Drag factor on asphalt = 0.80

d_G = Skid distance on grass/dirt = 8 ft

f_G = Drag factor on grass/dirt = 0.48

$$S = \sqrt{S_1^2 + 30 * d_A * f_A + 30 * d_G * f_G}$$

$$S = \sqrt{36.0^2 + 30 * 43 * 80 + 30 * 8 * 0.48}$$

$$S = \sqrt{1296 + 1032 + 115} = \sqrt{2443}$$

49.4 or 49 mph [80 kph]

Problem G.37

An automobile begins to yaw on a banked curve. The critical speed scuff mark laid down by the outside front tire was measured with a 60-ft [18.3 m] chord and found to have a middle ordinate of 21.5 in. [54.6 cm]. The LEVEL coefficient of friction is 0.72. Along the scuff mark the surface is banked 2%. Determine the speed of the vehicle.

Solution:

Solution of this problem has two main steps. We will ultimately utilize the critical speed to sideslip equation (banked surface). First we must determine the radius of curvature of the tire mark used to estimate the radius of curvature of the center of mass. The equation:

$$R = \frac{C^2}{8 * m_O} = \frac{m_O}{2}$$

where R = Radius of curvature, feet

C = Chord length = 60 ft

m_O = Middle ordinate length, ft = 21.5"/12 = 1.79 ft

Substituting and solving:

$$R = \frac{60^2}{8 * 1.79} + \frac{1.79}{2} = 252\,\text{ft}$$

We now plug the radius of curvature, lateral friction coefficient, and cross slope into the critical speed to sideslip formula. The equation:

$$S = \frac{\sqrt{15 * R * (f + m)}}{\sqrt{(1 - f * m)}}$$

where S = Speed in miles per hour

f = Lateral friction coefficient = .72

R = Radius of curvature = 252 ft

m = cross slope = 2% = 2/100 = 0.02

Substituting and solving:

$$S = \frac{\sqrt{15 * 252 * (0.72 + 0.02)}}{\sqrt{(1 - 0.72 * 0.02)}} = \frac{\sqrt{2797}}{\sqrt{0.986}}$$

$$S = 52.89/0.993 = 53.26 \text{ or } 53 \text{ mph [86 kph]}$$

PROBLEM G.38

According to the NHTSA NCAP crash test, a 2003 Toyota Avalon struck a full length rigid barrier at 35.2 mph [56.6 km/h] and ended up with 19.1 in. [48.5 cm] of crush (average) across the front. Using a threshold of measureable damage of 7 mph [11 km/h], determine the Campbell equation for the front of this vehicle. Calculate the 'A,' 'B,' and 'G' stiffness coefficients that would be used in the CRASH computer program for the truck's front. Use a damage width of 68 in. [173 cm] and a curb weight of 3387 lbs [1536 kg].

Solution:

The Campbell model, in equation form:

$$ebs = b_1 * C_{AVE} + b_0$$

where ebs = Equivalent barrier speed, mph = 35.2 mph

b_1 = Stiffness constant, mph/in.

C_{AVG} = Average crush depth, in = 19.1 in.

b_0 = Damage threshold constant, given 7 mph in this problem

The equation can also be generated for metric units. The b, and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph

$$ebs = b_1 * C_{AVG} + b_0$$

$$35.2 = b_1 * 19.1 + 7$$

$$28.2 = b_1 * 19.1$$

$$1.48 = b.$$

Therefore, the equation:

$$ebs = 1.48 * C_{AVG} + 7$$

The CRASH3 stiffness coefficients can also be derived for the Toyota. We will use curb weight (W = 3387 lb), the average crush, a 7 mph damage threshold, and the damage width (L = 68 in.).

Because we are using 7 mph as the damage threshold constant, the b_1 and b_0 constants will be the same, except that mph must be converted to inches per second:

$$b_0 = 7 \text{ mph} * \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in.}}{\text{ft}} = 123.2 \text{ in./s}$$

$$b_1 = \frac{1.48 \text{ mph}}{\text{in}} * \frac{1.467 \text{ ft/s}}{\text{mph}} * \frac{12 \text{ in.}}{\text{ft}} = 26.0/\text{s}$$

The CRASH3 constants:

$$A = \frac{W * b_0 * b_1}{g * L} = \frac{3387 * 123.2 * 26.0}{382.4 * 68} = 417$$

$$B = \frac{W * b_1 * b_1}{g * L} = \frac{3387 * 26.0 * 26.0}{382.4 * 68} = 88$$

$$G = A^2/2*B = 417^2/2*88 = 987$$

Note that 'g,' the acceleration of gravity, is given as (32.2*12) inches per second per second in order to make the units consistent.

Problem G.39

A passenger train is traveling at a constant rate of 74 mph [119 km/h]. When it is 2400 ft away from a grade crossing it activates the crossing warning lights. A car is approaching the crossing at a constant speed of 55 mph [88 km/h]. It tries to beat the train but is struck broadside by the train. How far was the car from the crossing when the signal lights activated.

Solution:

We have another time/distance problem on our hands. First we convert speeds in mph to velocities in feet per second.

The train's velocity:

$$V_T = S_T*1.467 = 74*1.467 = 108.5 \text{ ft/s}$$

The car's velocity:

$$V_C = S_C*1.467 = 55*1.467 = 80.7 \text{ ft/s}$$

Next, we determine how much time it took the train to go from the signal trip point to point of impact (2400 ft):

$$t = d/V_T = 2400/108.5 = 22.1 \text{ s}$$

When the train stated the warning signal, the car was

$$D_C = V_C*t = 80.7*22.1 = 1783 \text{ ft}$$

back from the point of impact. This distance in this case is measured from the point of impact on the cars structure.

Problem G.40

A train consists of 3 locomotives (weight: 140 tons 1127,000 kg] each), 42 loaded hopper cars (weight: 85 tons [77,000 kg] each), and 24 loaded flatcars (weight: 73 tons [66,000 kg] each). When the brake lines are fully pressurized, each locomotive generates 10,800 lb [48,000 kN] of stopping force, each hopper car generates 5900 lb [26,000 kN] of stopping force, and each flatcar generates 5200 lb [23,000 kN] of stopping force. Determine the deceleration of each locomotive and car type as if it were sliding independently. Determine the deceleration of the entire train.

Solution:

To solve this problem we use the basic physics equation drag factor (f) equals friction force (F) (sometimes called slopping force) divided by weight (W). For the equation to work, all units of measure must be consistent. Therefore, we will convert stopping force into tons. One ton equals 2000 lb.

Locomotive stopping force: 10,800 lb/2000 lb per ton = 5.4 tons

Hopper car stopping force: 5900 lb/2000 lb per ton = 2.95 long

Flat car stopping force: 5200 lb/2000 lb per ton = 2.6 tons

After drag factor for each car is determined, it can be multiplied by the acceleration of gravity (g), which at the earth's surface is 32.2 ft/s per second, to get each vehicle's deceleration.

Locomotive: $f = F/W = 5.4/140 = 0.0386$
$a = f*g = 0.0386*32.2 = 1.24 \text{ ft/s}^2$

Hopper car: $f = F/W = 2.92/85 = 0.0343$
$a = f*g = 0.0347*32.2 = 1.11 \text{ ft/s}^2$

Flat car: $f = F/W = 2.6/73 = 0.0356$
$a = f*g = 0.0356*32.2 = 1.15 \text{ ft/s}^2$

We do the same thing to get the deceleration of the entire train. But first, we must add up the total weights of each locomotive and car, and the total stopping force on each locomotive and car (3 locomotives, 42 hopper cars, 24 flat cars).

Train weight: $W = 3*140 + 42*85 + 24*33 = 5742$ tons

Train stopping force: $F = 3*5.4 + 42*2.95 + 24*2.6 = 202.5$ tons

Train drag factor: $f = F/W = 202.5/5742 = 0.0352$

Train deceleration: $a = f*g = 0.0352* 32.2 = 1.14 \text{ ft/s}^2$

Alternatively, we could have converted locomotive and car weights to pounds, divided that into stopping forces (in pounds) and gotten the same results.

Problem G.41

A 2003 Chevrolet Trailblazer, driven by an intoxicated driver, is approaching a T intersection. The driver fails to notice the stop sign in time. The truck brakes hard for 59 ft [18.0 m] on the asphalt road surface, 10 ft [3.0 m] grassy dirt surface, and collides head-on with a concrete retaining wall. The front of the truck is later measured and found to have an average crush of 21.7 in. [55 cm]. The drag factors were 0.80 for the asphalt surface and 0.48 for the dirt/grass surface. Estimate both the truck's impact speed and its speed at the start of precrash skid. The Trailblazer was crash tested at 29.7 and only suffered 9.0 in. of crush.

Solution:

We will derive the Campbell equation and then use it to estimate the impact speed into the brick wall. We will then use the combined speed formula to estimate speed at the start of skid. The Trailblazer was crash tested at 29.7 and only suffered 9.0 in. of crush.

The Campbell model, in equation form:

$$ebs = b_1*C_{AVG} + b_0$$

where ebs = Equivalent barrier speed = 29.7 mph

b_1 = Stiffness constant, mph/in.

C_{AVG} = Average crush depth = 9.0 in.

b_0 = Damage threshold constant, given 7 mph in this problem

The equation also can be generated for metric units. The b, and b_0 would have to be different than the imperial unit equation.

Assume $b_0 = 7$ mph

$$ebs = b_1 * C_{AVG} + b_0$$

$$29.7 = b_1 * 9.0 + 7$$

$$22.7 = b_1 * 9.0$$

$$2.52 = b_1$$

Therefore, the equation:

$$ebs = 2.52 * C_{AVG} + 7$$

Because the car hit a fixed object and stopped dead, ebs equals impact speed:

$$S_1 = 2.52 * C_{AVG} + 7 = 2.52 * 16.7 + 7 = 49.1 \text{ mph}$$

The speed at the start of skid can be solved is by using the combined speed formula. The variables:

S_1 = Speed of V-l at impact = 49.1 mph

d_A = Skid distance on asphalt = 59 ft

f_A = Drag factor on asphalt = 0.80

d_G = Skid distance on grass/dirt = 10 ft

f_G = Drag factor on grass/dirt = 0.48

$$S = \sqrt{S_1^2 + 30 * d_A * f_A + 30 * d_G * f_G}$$

$$S = \sqrt{49.1^2 + 30 * 59 * 0.80 + 30 * 10 * 0.48}$$

$$S = \sqrt{2411 + 1416 + 144} = \sqrt{3971}$$

63.0 or 63 mph [101 kph]

Problem G.42

a) 2900-lb [1315 kg] V-2 is stopped at a traffic signal when it is struck in the rear by 3330-lb [1510 kg] eastbound V-l. After impact V-l traveled east 53 ft [16.1 m] at an average drag factor of 0.38. After impact, V-2 traveled east 28 ft [8.5 m] at an average drag factor of 0.72. Calculate the impact speed of V-l.

b) For the collision in Problem 7, recalculate V-l's impact speed using dissipation of energy. V-l has an average of 7 in. [18 cm] of crush on its front. V-2 has an average of 24 in. [61 cm] of crush on the rear. Use the following Campbell equations:

S in mph, C_{AVG} in inches:	V-l front:	$ebs = 1.40 * C_{MAX} + 7$
	V-2 rear:	$ebs = 1.18 * C_{MAX} + 5$
S in kph, C_{AVG} in cm:	V-l front:	$ebs = 0.89 * C_{MAX} + 11$
	V-2 rear:	$ebs = 0.75 * C_{MAX} + 8$

Solution:

a) This is a simple application of the in-line (one dimensional) conservation of momentum formula. The variables:

S_1 = Speed of V-l at impact

S_2 = Speed of V-2 at impact = 0 mph

s_1 = Speed of V-l after impact

s_2 = Speed of V-2 after impact

d_1 = V-l post impact skid distance = 53 ft

d_2 = V-2 post impact skid distance = 28 ft

f_1 = Drag factor of V-l = 0.38

f_2 = Drag factor of V-2 = 0.72

W_1 = Weight of V-l = 3330 lb

W_2 = Weight of V-2 = 2900 lb

The weight ratio: If $W_1 = 1$, $W_2 = 2900/3330 = 0.879$

To find speeds after impact, we use the post impact skid distance and the basic skid formula. For Vehicle One:

$$s_1 = \sqrt{30 * d_1 * f_1} = \sqrt{30 * 53 * 0.38} = 24.6 \text{ mph}$$

Speed after impact of Vehicle Two:

$$s_2 = \sqrt{30 * d_2 * f_2} = \sqrt{30 * 34 * 0.72} = 24.6 \text{ mph}$$

The conservation of momentum formula:

$$S_1 * W_1 + S_2 * W_2 = s_1 * W_1 + s_2 * W_2$$

$$S_1 * 1 + 0 * 0.879 = 24.6 * 1 + 24.6 * 0.879$$

$$S_1 = 24.6 + 21.6$$

$$S_1 = 46.2 \text{ or } 46 \text{ mph [74 kph]}$$

b) Solution of this problem can be done in two steps. We will first calculate equivalent barrier speed for the damage on each vehicle. We will then use the dissipation of energy equation to solve for V-l's impact speed. Besides the data from the previous, we will use the following variables:

C_{AVG1} = Average crush depth to V-l front = 7 in.

C_{AVG2} = Average crush depth to V-2 rear = 24 in.

ebs_1 = Equivalent barrier speed of V-l

ebs_2 = Equivalent barrier speed of V-2

Equivalent barrier speed of V-l:

$$ebs_1 = 1.4{*}C_{AVG1} + 7 = 1.4{*}7 + 7 = 16.8$$

Equivalent barrier speed of V-2:

$$ebs_2 = 1.15{*}C_{AVG2} + 5 = 1.15{*}24 + 5 = 32.6$$

Substituting and solving the dissipation of energy equation:

$$S_1^2{*}W_1 + S_2^2{*}W_2 = s_1^2{*}W_1 + s_2^2{*}W_2 + ebs_1^2{*}W_1 + ebs_2^2{*}W_2$$

$$S_1^2{*}1 + 0^2{*}0.879 = 24.6^2{*}1 + 24.6^2{*}0.879 + 16.8^2{*}1 + 32.6^2{*}0.879$$

$$S_1^2 = 605 + 532 + 282 + 934 = 2352$$

$$S_1 = \sqrt{2352} = 48.51 \text{ or } 49 \text{ mph [78 kph]}$$

Problem G.43

For the corresponding vehicles in Problem 6, a Federal NCAP crash (Head-on into full length rigid barrier) for the Nissan showed that at 35.0 mph the vehicle suffered an average 18.3 in. of crush [46.5 cm @ 56.3 kph]. The NCAP test of the Ford shows it suffered 20.6 in. of crush at 34.8 mph [52.3 cm at 56.0 kph]. In the collision the vehicles suffered the following damage:

	Nissan	**Ford**
Damage Width	**Entire front**	**Entire front**
Crush Pt. C_1	18 in. [46 cm]	25 in. [63 cm]
Crush Pt. C_2	17.5 in. [44 cm]	23 in. [58 cm]
Crush Pt. C_3	17 in. [43 cm]	21.5 in. [55 cm]
Crush Pt. C_4	16 in. [41 cm]	20.5 in. [52 cm]
Crush Pt. C,	15.5 in. [39 cm]	20 in. [51 cm]
Crush Pt. C_6	13 in. [33 cm]	17 in. [43 cm]

Does the vehicle crush corroborate or contradict the EDR delta V reading?

Solution:

We were given that in a full length rigid barrier crash test, the Nissan was subjected to a 35.0 mph impact and suffered 18.3 in, of crush. We also were advised that the Ford was subjected to a 34.8 mph crash test and suffered 20.6 in, of crush. We now derive the Campbell equations for both vehicles:

For the Nissan: $ebs = b_1{*}C + b_0$

Assume $b_0 = 7$ mph $35.0 = b_1{*}18.3 + 7$

$$28.0 = b_1{*}18.3$$

$$1.53 = b_1$$

Therefore, the equation for the Nissan:

$$ebs_2 = 1.53{*}C_{AVG1} + 7$$

For the Ford: $ebs = b_1 * C_{AVG2} + b_0$

Assume $b_0 = 7$ mph $34.8 = b_1 * 20.6 + 7$

$27.8 = b_1 * 20.6$

$1.35 = b_1$

Therefore, the equation for the Ford:

$$ebs_2 = 1.35 * C_{AVG2} + 7$$

We will ultimately solve this problem using conservation of momentum and dissipation of energy. The parameters:

S_1 = Speed of the Nissan at impact

s_1 = Speed of the Nissan after impact = 0 mph

S_2 = Speed of the Ford at impact

s_2 = Speed of the Ford after impact = 0 mph

W_2 = Weight of the Nissan = 3426 lb

W_1 = Weight of the Ford = 4075 lb.

C_{AVG1} = Average crush depth to Nissan front

C_{AVG2} = Average crush depth to Ford front

ebs_1 = Equivalent barrier speed of Nissan

ebs_2 = Equivalent barrier speed of Ford

The weight ratios: If $W_1 = 1$, then $W_2 = 4075/3426 = 1.189$

We must estimate the energies dissipated by crush via determination of their equivalent barrier impact speeds. Average crush to the front of vehicle one:

$$C_{AVG1} = (C_1/2 + C_2 + C, + C_4 + C, + C_6/2)/5$$

$$C_{AVG1} = (18/2 + 17.5 + 17 + 16 + 15.5 + 13/2)/5 = 16.3 \text{ in.,}$$

Equivalent barrier speed:

$$ebs_1 = 1.53 * C_{AVG1} + 7 = 1.53*16.3 + 7 = 31.9 \text{ mph}$$

Average crush to the front of Vehicle Two:

$$C_{AVG2} = (25/2 + 23 + 21.5 + 20.5 + 20 + 17/2)/5 = 21.2 \text{ in.,}$$

Crush energy equivalent speed:

$$ebs_2 = 1.35*C_{AVG2} + 7 = 1.35*21.2 + 7 = 35.6 \text{ mph}$$

We now substitute into the in-line conservation of momentum formula. Note that term for V-2's impact momentum is negative because it is heading in the opposite direction of V-l.

$$S_1*W_1 + S_2*W_2 = s_1*W_1 + s_2*W_2$$

$$S_1*1 - S_2*1.189 = 0*1 + 0*1.189$$

$$S_1 - S_2*1.189 = 0 \qquad S_1 = S_2*1.189$$

Substituting and solving the dissipation of energy equation:

$$S_1^2*W_1 + S_1^2*W_2 = s_1^2*W, + s_2^2*W_2 + ebs_1^2*W_1 + ebs_2^2*W_2$$

$$S_1^2*1 + S_2^2*1.189 = 0^2*1 + 0^2*1.189 + 31.9^2*1 + 35.6^2*1.189$$

$$S_1^2 + S_2^2*1.189 = 1018 + 1507 = 2525$$

Substituting $S_1*1.189$ for S_1:

$$(1.189*S_2)^2 + S_2^2*1.189 = 2525$$

$$1.414*S_2^2 + S_2^2*1.189 = S_2^2*2.603 = 2525$$

$$S_2 = \sqrt{970} = 31.1 \text{ or } 31 \text{ mph [50 kph]}$$

Substituting this into the momentum equation:

$$S_1 = S_2*1.189 = 31.1*1.189 = 36.98 \text{ or } 37 \text{ mph [60 km/h]}$$

The impact speeds calculated by energy/momentum method are reasonably consistent with the Ford's EDR reading.

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Some of the exercises in this book *Road Vehicle Dynamics: Problems and Solutions* refer to specific material that is published in the companion volume *Road Vehicle Dynamics* by Rao Dukkipati, Jian Pang, Mohamad Qatu, Gang Sheng, and Shuguang Zuo (Product Code R-366, SAE International, Warrendale, PA, 2008, ISBN 978-0-7680-1643-7). Although this book contains adequate material to stand alone and provide valuable hands-on experience in solving road vehicle dynamics problems, readers are urged to purchase or refer to the companion book, *Road Vehicle Dynamics*, to obtain the fullest benefit from all of these exercises.

Road Vehicle Dynamics

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Mohamad S. Qatu
Gang Sheng
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